Learning Objective 3.8: I can identify grating lobes, beam squint, and quantization effects in a phased array and describe their impact on system performance.

Show your work. A **Documentation** line at the end is required (write **None** if you did not collaborate).

1. **Where the array factor repeats.** The ADALM-PHASER carries 8 patch elements on a 14 mm pitch. A thinning demonstration feeds only every third element, so the fed elements sit 42 mm apart. Work at the array design center, 10.3 GHz, where $\lambda = 29.1$ mm.

   (a) In one or two sentences, explain why a grating lobe reaches the same height as the main lobe, while a sidelobe does not.

   At a grating-lobe angle the path-length difference between neighboring elements is a whole wavelength rather than zero, so every element contribution arrives in phase exactly as it does at the commanded angle. The array cannot distinguish the two directions, and the summation reaches its maximum value N in both. A sidelobe is a partial cancellation of that same summation, so it is always below the peak.

   (b) With the beam at broadside, find every grating-lobe angle in visible space.

   θ_g = $\pm 43.9^\circ$

   $$\sin\theta_g = \sin\theta_0 \pm m\frac{\lambda}{d} = 0 \pm m\frac{29.1}{42}$$
   $$= \pm 0.693m$$

   For $m = 1$, $\theta_g = \arcsin(\pm 0.693) = \pm 43.9^\circ$. For $m = 2$ the required sine is ± 1.386, which is not a real angle, so the pattern has three full-height beams and no more.

   (c) Now steer the beam to $\theta_0 = +20^\circ$. Find every grating lobe that remains in visible space.

   θ_g = -20.5°

   $$\sin\theta_g = \sin(20^\circ) \pm 0.693 = 0.342 \pm 0.693$$

   The plus sign gives 1.035, which is outside $[-1, 1]$, so that lobe has left visible space. The minus sign gives -0.351, so

   $$\theta_g = \arcsin(-0.351) = -20.5^\circ$$

   Steering the main beam to $+20^\circ$ pushed one grating lobe past the horizon and pulled the other in from -43.9° to -20.5°. Grating lobes move uniformly in $\sin\theta$, not in degrees.

   (d) A classmate proposes applying a Hann taper to remove the extra beams. Explain in one or two sentences why that does not work.

   A taper controls the sidelobe skirts that surround a single beam by smoothing the aperture illumination. A grating lobe is a second main beam produced by the element spacing, and it is present with any amplitude distribution, including a uniform one. The taper lowers the sidelobes around both beams equally and leaves both peaks at full height. The only cure is closer element spacing.

2. **Choosing the element spacing.** You are laying out an X-band linear array. Use $c = 3 \times 10^8$ m/s throughout.

(a) State the criterion that keeps grating lobes out of visible space over a scan range $\pm\theta_0$, and explain why half-wavelength spacing is the usual default.

The criterion is

$$d < \frac{\lambda}{1 + |\sin\theta_0|}$$

Taking the scan range to its limit, $\theta_0 \to 90°$, gives $d < \lambda/2$. A half-wavelength lattice therefore has no grating lobe at any commandable scan angle, so it is the spacing that keeps grating lobes out at every scan angle rather than tying the array to one scan sector.

(b) The array must scan to $\pm 60°$ and operate up to 10.5 GHz. Find the largest usable element spacing in millimeters.

Use the highest frequency, where λ is shortest and the criterion is tightest:

$$\lambda = \frac{3 \times 10^8}{10.5 \times 10^9} = 28.6 \text{ mm}$$

$$d < \frac{28.6}{1 + \sin(60°)} = \frac{28.6}{1.866} = 15.3 \text{ mm}$$

$d_{\max}$ = **15.3 mm**

(c) A competing layout uses $d = 20$ mm. At 10.5 GHz, how far off broadside can that array scan before a grating lobe enters visible space?

$$\frac{\lambda}{d} = \frac{28.6}{20} = 1.429$$

$$1 + |\sin\theta_0| < 1.429 \quad \Rightarrow \quad |\sin\theta_0| < 0.429$$

$$\theta_0 < \arcsin(0.429) = 25.4°$$

$\theta_{0,\max}$ = **25.4°**

(d) The PHASER uses $d = 14$ mm. State its scan limit from the same criterion, and give one sentence on what the design gives up by choosing a spacing this tight.

At 10.5 GHz, $\lambda/d = 28.6/14 = 2.04$, so the criterion requires $1 + |\sin\theta_0| < 2.04$, which every real angle satisfies. The array has no grating lobe at any scan angle in its band.

What it gives up is aperture. With the element count fixed at 8, tighter spacing means a shorter array, $L = Nd = 112$ mm instead of 160 mm at 20 mm pitch, so the beam is wider and the directivity lower. For a fixed element count, wide scan coverage and a narrow beam are competing goals.

3. **Beam squint across a band.** An 8-element array with $d = 14$ mm operates from 10.0 to 10.5 GHz. The beam-steering computer sets the element phases for $\theta_0 = 45°$ at the band center, $f_0 = 10.25$ GHz. Use $c = 3 \times 10^8$ m/s.

   (a) Explain in one or two sentences why a broadside beam does not squint at all.

   At broadside the commanded phase ramp is zero, because every element is equidistant from a source on boresight. Squint arises when a signal at a different wavenumber meets a ramp set for f_0, and scaling a ramp of zero still gives zero. The condition $\sin\theta = (f_0/f)\sin\theta_0$ returns $\theta = 0$ for any frequency.

   (b) Find the beam angle at the low band edge, 10.0 GHz.

$$\sin\theta = \frac{f_0}{f}\sin(45°) = \frac{10.25}{10.00}(0.7071)$$
$$= 1.025(0.7071) = 0.7248$$

$$\theta = \arcsin(0.7248) = 46.5°$$

   $\theta =$ $46.5°$

   The beam has squinted $+1.5°$ toward the horizon.

   (c) Find the beam angle at the high band edge, 10.5 GHz, and state the total angular walk of the beam across the band.

$$\sin\theta = \frac{10.25}{10.50}(0.7071) = 0.9762(0.7071) = 0.6903$$

$$\theta = \arcsin(0.6903) = 43.6°$$

   The total walk from one band edge to the other is

$$46.45° - 43.65° = 2.80°$$

   walk = $2.8°$

(d) The half-power beamwidth of this array at 45° is $\theta_{\text{HP}} \approx 0.886\lambda/(Nd\cos(\theta_0))$. Express the walk from part (c) as a fraction of a beamwidth, and recommend a fix if the requirement were to hold the walk under 2% of a beamwidth.

walk / θ_{HP} = 0.15

At 10.25 GHz, $\lambda = 29.3$ mm and $Nd = 112$ mm:

$$\theta_{\text{HP}} = \frac{0.886(29.3)}{112\cos(45^\circ)} = \frac{25.9}{79.2} = 0.327 \text{ rad} = 18.8^\circ$$

$$\frac{2.8^\circ}{18.8^\circ} = 0.15$$

The beam walks about 15% of a beamwidth, which loses a few tenths of a decibel at the aimpoint and is usually acceptable for a narrowband link. Holding it under 2% is a factor of seven, and no phase setting achieves it because the fractional bandwidth and $\tan(\theta_0)$ set the walk. The fix is to steer with true time delay instead of phase, at the element or at the subarray level, so that the delay compensates the path difference itself and every frequency in the band points the same way.

4. **Phase-Shifter Resolution.** The ADAR1000 beamformer on the PHASER uses a B-bit phase shifter. Work at the HB100 source frequency, 10.525 GHz, where $\lambda = 28.5$ mm and $d/\lambda = 0.491$.

   (a) Find the phase step (LSB) for $B = 6$ and for the ADAR1000's $B = 7$.

   $$\text{LSB} = \frac{360^\circ}{2^B}$$

   $$B = 6: \quad \frac{360^\circ}{64} = 5.625^\circ$$

   $$B = 7: \quad \frac{360^\circ}{128} = 2.8125^\circ$$

   $B = 6$: 5.625°

   $B = 7$: 2.8125°

   (b) With $B = 7$, find the pointing step near broadside and at $\theta_0 = 60^\circ$.

   The inter-element phase is $\Delta\phi = 360^\circ(d/\lambda)\sin\theta_0$, so a change of one LSB moves $\sin\theta_0$ by

   $$\Delta(\sin\theta_0) = \frac{\text{LSB}}{360^\circ(d/\lambda)} = \frac{2.8125}{360(0.491)} = 0.0159$$

   Near broadside $\sin\theta_0 \approx \theta_0$ in radians, so

   $$\Delta\theta_0 = 0.0159 \text{ rad} = 0.91^\circ$$

   Off broadside divide by $\cos\theta_0$:

   $$\Delta\theta_0 = \frac{0.0159}{\cos(60^\circ)} = 0.0318 \text{ rad} = 1.82^\circ$$

   Both are small next to the array's 13.2° beamwidth.

   broadside: 0.91°

   at 60°: 1.82°

   (c) Use the rule of thumb QSLL $\approx -6B$ dB for $B = 3$ and $B = 7$, and state which sidelobe mechanism dominates in each case for a uniform 8-element array.

   $$B = 3: \quad \text{QSLL} \approx -18 \text{ dB}$$

   $$B = 7: \quad \text{QSLL} \approx -42 \text{ dB}$$

   A uniform 8-element array has its own first sidelobe at about -13 dB. At three bits the quantization lobes are already below that, and at seven bits they are far below it, so in both cases the ordinary sidelobe structure of the uniform illumination dominates. Quantization only takes over below about two bits, where the rule predicts -12 dB and the array's own sidelobes are no longer the largest feature. The hardware disagrees at three bits: phaser-00 measured a -12 dBc quantization lobe there, level with the -13 dB sidelobes.

   $B = 3$: -18 dB

   $B = 7$: -42 dB

(d) A classmate argues that because the phase shifter resolves only $2.8125°$, a pattern null on the PHASER can never go much deeper than 20 dB. Explain in two or three sentences why that reasoning is wrong, and state what does set the measured depth.

Rounding the null weights onto the $2.8125°$ phase grid, with gain steps of about 1%, still produces a notch near -48 dB at the angle the weights were designed for. Quantization displaces the notch by a fraction of a degree from its commanded angle rather than filling it in, so it is a pointing error and not a depth limit.

What sets the measured depth is the floor of the sweep and the bench around the kit. On phaser-00 the L24 notch measured -46 to -50 dB, close to the prediction, while the same kit's L22–L23 sweeps read floors from -17 to -31 dBc, so the depth you see depends on the room as much as on the weights.

5. **Reading a sweep and sizing a design.** Each part below reports something a student measured on the PHASER, or asks for a design decision. The array is 8 elements on a 14 mm pitch unless stated otherwise.

   (a) A broadside sweep against the HB100 at 10.525 GHz ($\lambda = 28.5$ mm) shows three peaks of equal height, at 0° and $\pm 42^\circ$. What effective element spacing does that indicate, and how were the elements fed?

   d_{eff} = 42.6 mm

   $$\sin\theta_g = \frac{\lambda}{d} \quad \Rightarrow \quad d = \frac{\lambda}{\sin(42^\circ)} = \frac{28.5}{0.669} = 42.6 \text{ mm}$$

   That is three times the 14 mm pitch, so the student fed every third element and switched off the other five. Equal peak heights confirm grating lobes rather than sidelobes.

   (b) The student commands the array to $+45^\circ$ with phases set at 10.525 GHz, but the sweep peaks at $+47.9^\circ$ with the source running at 10.025 GHz. Name the effect and confirm the measurement.

   θ = 47.9°

   This is beam squint: the software computed the phase ramp at one frequency, and the signal arrived at another.

   $$\sin\theta = \frac{10.525}{10.025}\sin(45^\circ) = 1.0499(0.7071) = 0.7424$$

   $$\theta = \arcsin(0.7424) = 47.9^\circ$$

   The measurement matches the prediction, so nothing is wrong with the array.

   (c) At $B = 2$ the sweep shows a lobe near -55° about 7 dB below the peak, while the rule of thumb predicts -12 dB. Explain the discrepancy in one or two sentences.

   The $-6B$ figure is an RMS estimate derived for a large aperture, where the staircase phase error repeats over many periods and the resulting lobes average out to the predicted level. An 8-element array holds only a couple of periods of that sawtooth, so individual lobes scatter several decibels either side of the estimate. The rule gives the expected level, not a bound on any single lobe.

(d) A new array must scan to $\pm 50^\circ$ over 10.0 to 10.5 GHz and hold every sidelobe below -25 dB. State the maximum element spacing and the minimum number of phase bits, and say which of the three defects the bandwidth requirement drives.

d_{max} = 16.2 mm

B_{min} = 5 bits

Spacing, evaluated at the top of the band where $\lambda = 28.6$ mm:

$$d < \frac{28.6}{1 + \sin(50^\circ)} = \frac{28.6}{1.766} = 16.2 \text{ mm}$$

Bits, from the quantization sidelobe rule:

$$-6B < -25 \quad \Rightarrow \quad B > 4.2 \quad \Rightarrow \quad B = 5$$

The ADAR1000's seven bits clear this with margin. The bandwidth requirement drives beam squint: the band is 4.9% wide, so at 50° the beam walks roughly $(0.024)\tan(50^\circ) = 0.029$ rad $= 1.7^\circ$ either side of center. That is a small fraction of the beamwidth here, so phase steering is adequate; a wider band or a wider scan would force true time delay.

Documentation: _____