Learning Objective 3.2: I can derive the array factor for an arbitrary linear array and apply pattern multiplication.

Show your work. A **Documentation** line at the end is required (write **None** if you did not collaborate).

1. **Beamwidth of the course array.** The ADALM-PHASER array has $N = 8$ patch elements on $d = 14$ mm centers. For this problem, the source is at 10.3 GHz. Use $\theta_{\text{HP}} \approx 0.886\lambda/\left(Nd\cos\theta_0\right)$ throughout.

   (a) Find the wavelength and the element spacing in wavelengths.

   $$\lambda = \frac{3\times10^8}{10.3\times10^9} = 29.1 \text{ mm}$$
   $$\frac{d}{\lambda} = \frac{14}{29.1} = 0.481$$

   The aperture is $Nd = 112$ mm $= 3.85\lambda$.

   d/λ = **0.481**

   (b) Compute the half-power beamwidth at broadside.

   $$\theta_{\text{HP}} = \frac{0.886}{3.85} = 0.230 \text{ rad}$$
   $$= 13.2^\circ$$

   θ_{HP} = **13.2°**

   (c) Compute the half-power beamwidth with the beam steered to $\theta_0 = 45^\circ$.

   $$\theta_{\text{HP}} = \frac{13.2^\circ}{\cos\left(45^\circ\right)} = \frac{13.2^\circ}{0.707} = 18.7^\circ$$

   θ_{HP} = **18.7°**

   (d) In one or two sentences, explain physically why the beam widens by $1/\cos\theta_0$ when the array is steered, even though no hardware changed.

   A source in the direction θ_0 sees the aperture foreshortened: the projected length of the array normal to the beam is $Nd\cos\theta_0$, not Nd. Beamwidth is set by the aperture in wavelengths, so the effective aperture shrinks as the beam steers away from broadside and the beam widens by the same factor. At 60° the projected aperture is half its broadside value.

2. **Nulls and the first-null beamwidth.** Same array as Question 1: $N = 8$, $d/\lambda = 0.481$, so $Nd = 3.85\lambda$. The nulls of a uniform array are at $\sin\theta = \sin\theta_0 \pm m\lambda/(Nd)$ for $m = 1, 2, 3, \ldots$, excluding multiples of N.

   (a) With the beam at broadside, find every null angle on the positive side of the pattern.

   The tooth spacing in sine space is $\lambda/Nd = 1/3.85 = 0.260$.

   nulls = $15.1, 31.3, 51.2^\circ$

   $$m = 1: \quad \sin(\theta) = 0.260 \rightarrow 15.1^\circ$$
   $$m = 2: \quad \sin(\theta) = 0.520 \rightarrow 31.3^\circ$$
   $$m = 3: \quad \sin(\theta) = 0.780 \rightarrow 51.2^\circ$$
   $$m = 4: \quad \sin(\theta) = 1.040 \rightarrow \text{no real angle}$$

   (b) Compute the first-null beamwidth at broadside, and the ratio of it to the half-power beamwidth from Question 1(b).

   FNBW = 30.1°

   $$\text{FNBW} = 2\arcsin\left(\frac{\lambda}{Nd}\right) = 2(15.1^\circ) = 30.1^\circ$$
   $$\frac{\text{FNBW}}{\theta_{\text{HP}}} = \frac{30.1}{13.2} = 2.3$$

   (c) The array factor of an 8-element array has 7 nulls per period, but part (a) found only 3 on each side of broadside. Explain in one or two sentences where the others went.

   The nulls are equally spaced in $\sin\theta$, and only those landing inside the visible region $|\sin\theta| \leqslant 1$ correspond to real directions. With $\lambda/Nd = 0.260$, the fourth null would require $\sin\theta = 1.04$, which is outside visible space. The pattern is periodic in sine space, but the antenna can only radiate into the visible part of it.

   (d) Steer the beam to $\theta_0 = 30^\circ$ and find the two nulls that bracket the main lobe, then the first-null beamwidth.

   FNBW = 35.6°

   $$\sin(\theta) = 0.500 - 0.260 = 0.240 \rightarrow 13.9^\circ$$
   $$\sin(\theta) = 0.500 + 0.260 = 0.760 \rightarrow 49.5^\circ$$
   $$\text{FNBW} = 49.5^\circ - 13.9^\circ = 35.6^\circ$$

   The peak is 16.1° above the lower null and 19.5° below the upper one, so a steered beam is not symmetric about its own peak.

3. **Directivity of a uniform linear array.** Use $D \approx 2Nd/\lambda$ for a uniform broadside array of isotropic elements.

(a) Compute the directivity of the 8-element course array, in linear units and in decibels.

$D =$ $7.7 = 8.9$ dB

$$D \approx 2\,(8)\,(0.481) = 7.7$$
$$D_{\text{dB}} = 10\log_{10}(7.7) = 8.9 \text{ dB}$$

(b) Suppose the same 8 elements were half-wave dipoles rather than patches. Estimate the total gain of the array in dBi.

$G_{\text{total}} =$ 11.0 dBi

Pattern multiplication becomes addition in decibels:

$$G_{\text{total}} = G_{\text{element}} + D_{\text{array}}$$
$$= 2.15 \text{ dBi} + 8.86 \text{ dB} = 11.0 \text{ dBi}$$

This holds while the element pattern is nearly flat across the main lobe.

(c) Turning off the outer four elements leaves the center four running. State what happens to the directivity and to the half-power beamwidth, with numbers, and explain in one sentence why both changes have the same cause.

The aperture halves from 3.85λ to 1.92λ. Directivity halves, $D \approx 3.85 = 5.9$ dB , a drop of exactly 3 dB , and the beamwidth doubles from 13.2° to about 27°. Both follow from the aperture in wavelengths: directivity is proportional to Nd/λ and beamwidth is inversely proportional to it, so any change in aperture moves them in opposite directions by the same factor.

4. **Sizing an array.** A ground-based X-band tracker at 10 GHz requires a half-power beamwidth of 2.0° and must steer to $\pm 60^\circ$.

   (a) Find the aperture length required at broadside, in millimeters and in wavelengths.

   $Nd =$ $762 \text{ mm} = 25.4\lambda$

   $$\lambda = 30 \text{ mm}, \qquad \theta_{\text{HP}} = 2.0^\circ = 0.0349 \text{ rad}$$
   $$Nd = \frac{0.886\lambda}{\theta_{\text{HP}}} = \frac{0.886\,(30)}{0.0349} = 762 \text{ mm}$$
   $$\frac{Nd}{\lambda} = 25.4$$

   (b) The grating-lobe criterion is $d < \lambda/(1 + |\sin\theta_0|_{\max})$. Find the largest usable spacing for this scan volume, then the element count if you build the array at $d = \lambda/2$.

   $d_{\max} =$ 0.536λ

   $N =$ 51

   $$d < \frac{\lambda}{1 + \sin(60^\circ)} = \frac{\lambda}{1.866} = 0.536\lambda = 16.1 \text{ mm}$$
   $$N = \frac{762}{15} = 50.8 \rightarrow 51 \text{ elements}$$

   Round up, never down: rounding down misses the beamwidth specification.

   (c) A colleague proposes $d = 0.9\lambda$ instead, which reaches the same 25.4λ aperture with 28 elements rather than 51 — a large saving in phase shifters and amplifiers. Evaluate the proposal: compute where the grating lobe lands at the edge of the scan volume, and state whether you would accept it.

   Grating lobes appear at $\sin\theta_g = \sin\theta_0 \pm \lambda/d$. With $d = 0.9\lambda$, $\lambda/d = 1.111$. At broadside, $\sin\theta_g = 1.111$ is outside visible space and nothing is wrong. Steered to 60°:

   $$\sin(\theta_g) = 0.866 - 1.111 = -0.245 \rightarrow -14.2^\circ$$

   A full-strength second beam appears at -14.2° whenever the array looks at the edge of its volume, so the tracker would report targets in the wrong direction. Reject the proposal: the saving in elements is real, but the array no longer meets the scan requirement.

   (d) The beamwidth requirement must hold across the whole scan volume, not just at broadside. State the aperture this demands and the resulting element count at $d = \lambda/2$.

   The beam is widest at the edge of the volume, so size for $\theta_0 = 60^\circ$:

   $$Nd = \frac{762}{\cos(60^\circ)} = 1524 \text{ mm} = 50.8\lambda, \qquad N = \frac{1524}{15} = 102$$

   The last 60° of scan doubles the array. Whether the extra elements are worth their cost is a system question, and we settle it before we fix the aperture.

5. **Reading a measured sweep.** A student sweeps the course array at 10.3 GHz ($\lambda = 29.1$ mm) with the GUI and records a main lobe 13.1° wide at half power, a null-to-null width of 31.0°, and a first sidelobe at -11.5 dBc.

   (a) From the measured half-power width alone, infer the aperture in millimeters and the number of active elements on 14 mm centers.

   $N =$ 8

   $$Nd = \frac{0.886\lambda}{\theta_{\text{HP}}} = \frac{0.886\,(29.1)}{0.2287} = 113 \text{ mm}$$
   $$N = \frac{113}{14} = 8.1 \rightarrow 8 \text{ elements}$$

   (b) Theory puts the first-null beamwidth at 30.1° and the student measured 31.0°. The sweep steps the element-to-element phase by the phase-shifter resolution of 2.8125° per point. Explain the discrepancy in one or two sentences.

   One phase step moves $\sin\theta$ by $2.8125^\circ / (360^\circ \times 0.481) = 0.0162$, which is 0.93° of angle near broadside. The recorded trace almost never lands on the exact null at 15.1°, and reading each null at its lowest sample places it up to one step outside the true null, so the measured width falls between 30.1° and about 31.1°. The 0.9° difference comes from the sampling grid, not from the array.

   (c) The measured first sidelobe is -11.5 dBc, while theory for a uniform 8-element array gives -12.8 dBc. Is this evidence of a fault? Answer in one or two sentences.

   No. On this hardware we expect a uniform array's first sidelobe between -11 and -13 dBc once we include element-to-element amplitude and phase errors, the coarse sweep grid, and the finite noise floor, and -11.5 dBc sits inside that band. A sidelobe several decibels high, or a main lobe that is not at the commanded angle, would be evidence of a fault.

   (d) A second sweep on the same kit, with a different set of element gains loaded, shows a main lobe 27° wide with no visible sidelobes above the noise floor. State which aperture configuration this is and what it implies about the number of active elements.

   $N =$ 4

   $$\theta_{\text{HP}} = 27^\circ = 0.471 \text{ rad}$$
   $$Nd = \frac{0.886\,(29.1)}{0.471} = 54.7 \text{ mm} \approx 1.9\lambda$$

   That is four elements at 14 mm spacing, so the center four are on and the outer four are off. With only four elements the sidelobes rise to about -11 dBc, so "none visible" points instead to a tapered set of element gains rather than a clean four-element aperture.

Documentation: _____