Learning Objective 3.2: I can derive the array factor for an arbitrary linear array and apply pattern multiplication.

Show your work. A **Documentation** line at the end is required (write **None** if you did not collaborate).

1. **Building the sum.** Three isotropic elements sit on a line at $z = 0$, $z = d$, and $z = 2d$ with $d = \lambda/2$. They are fed in phase with amplitude weights $a_0 = 1$, $a_1 = 2$, $a_2 = 1$. We measure the scan angle θ from broadside.

   (a) Pattern multiplication lets you write the array pattern as one element pattern times one sum. State the two conditions the elements must satisfy for that factoring to be valid, and name the step in the derivation that uses each one.

   The elements must be **identical** and **identically oriented**. The superposition step uses both: only if every element carries the same $E_0(\theta)$ can we pull that factor outside the sum. If the elements differ, or one is rotated, each term keeps its own pattern.

   (b) Write the array factor as a function of $\psi = kd\sin(\theta)$, then find the extra path length and the extra phase that element 2 contributes relative to element 0 when $\theta = 30^\circ$.

   $\Delta r_2 =$ 0.5λ

   $$AF(\psi) = 1 + 2e^{j\psi} + e^{j2\psi}$$

   $\Delta\phi_2 =$ 180°

   The extra path is the projection of the element position on the look direction:

   $$\Delta r_2 = 2d\sin(30^\circ) = 2(0.5\lambda)(0.5) = 0.5\lambda$$

   $$\Delta\phi_2 = k\,\Delta r_2 = \frac{2\pi}{\lambda}(0.5\lambda) = \pi = 180^\circ$$

   (c) Evaluate $|AF|$ at broadside and at $\theta = 30^\circ$, and give the 30° value in dB relative to the broadside peak.

   Factor the sum:

   $|AF|_{\theta=0} =$ 4

   $$AF = e^{j\psi}(2 + 2\cos\psi) \quad\Longrightarrow\quad |AF| = 2 + 2\cos\psi$$

   Level at $30^\circ =$ -6.0 dB

   $$\theta = 0: \quad \psi = 0, \qquad |AF| = 4$$

   $$\theta = 30^\circ: \quad \psi = \pi\sin(30^\circ) = 90^\circ$$

   $$|AF| = 2 + 2\cos(90^\circ) = 2$$

   $$20\log_{10}(2/4) = -6.0 \text{ dB}$$

   (d) The derivation keeps the phase difference between elements but throws away the amplitude difference. Explain in one or two sentences why that is consistent, given a 112 mm array observed at a range of 30 m.

The path lengths differ by at most the array length, 112 mm, which against a 30 m range changes $1/r$ by about 0.4%. That same 112 mm is nearly four wavelengths at X-band, so it changes the phase by up to 1400°, so the pattern depends on the phase differences alone.

2. **The course array.** The PHASER carries $N = 8$ patch elements on a line with spacing $d = 14$ mm, uniformly excited and unsteered. Work at $f = 10.3$ GHz.

   (a) Find λ, the spacing in wavelengths, and the range of the inter-element phase ψ over the visible region.

   $$\lambda = \frac{3 \times 10^8}{10.3 \times 10^9} = 29.1 \text{ mm}$$

   $$\frac{d}{\lambda} = \frac{14}{29.1} = 0.481$$

   d/λ = 0.481

   ψ range = $\pm 173^\circ$

   At broadside $\psi = kd\sin(\theta)$, and $\sin(\theta)$ runs over ± 1:

   $$\psi_{\max} = kd = 2\pi(0.481) = 3.02 \text{ rad} = 173^\circ$$

   so $-173^\circ \leqslant \psi \leqslant +173^\circ$, just short of one full period.

   (b) Locate every null of the array factor that falls in the visible region.

   $$\sin(\theta_m) = \frac{m\lambda}{Nd} = \frac{m}{8(0.481)} = \frac{m}{3.85}$$

   $$m = 1: \ \theta = \arcsin(0.260) = 15.1^\circ$$

   $$m = 2: \ \theta = \arcsin(0.520) = 31.3^\circ$$

   $$m = 3: \ \theta = \arcsin(0.780) = 51.2^\circ$$

   $m = 4$ needs $\sin(\theta) = 1.04$, which is not a real angle.
   Nulls occur at $\pm 15.1^\circ$, $\pm 31.3^\circ$, $\pm 51.2^\circ$.

   Nulls = $\pm 15.1^\circ, \ \pm 31.3^\circ, \ \pm 51.2^\circ$

   (c) Find the first-null beamwidth and the half-power beamwidth, and state the first sidelobe level you expect.

   $$\text{FNBW} = 2\arcsin\left(\frac{\lambda}{Nd}\right) = 2(15.1^\circ) = 30.1^\circ$$

   $$\text{HPBW} \approx \frac{0.886\lambda}{Nd} = \frac{0.886}{3.85} = 0.230 \text{ rad} = 13.2^\circ$$

   Uniform excitation puts the first sidelobe about 13 dB down; the exact value for $N = 8$ is -12.8 dB , at $\pm 21.9^\circ$.

   FNBW = 30.1°

   HPBW = 13.2°

   SLL = -12.8 dB

   (d) A uniform array of N elements has $N - 2$ sidelobes in one period of ψ. Count the sidelobes that fall in the visible region for this array, and explain the count using your answer to part (a).

   With $N = 8$ there are six sidelobes per period, three on each side of the main lobe. The visible region spans $\pm 173^\circ$ of ψ, which is 96% of the $\pm 180^\circ$ period, so all six appear — the three nulls

found in part (b) bracket three sidelobes on each side. Nothing is cut off because the spacing is just under half a wavelength.

3. **Pattern multiplication.** Four short dipoles lie end to end along the array axis, spaced $d = \lambda/2$ and fed in phase. In scan angle the short dipole's element factor is $EF(\theta) = \cos(\theta)$. The four-element uniform array factor has its first sidelobe at $\theta = 47.1°$, where $AF_4 = -11.3$ dB .

   (a) Evaluate the element factor at the sidelobe angle in dB, and use it to estimate the first sidelobe level of the total pattern.

   EF at $47.1°$ = -3.3 dB

   Total SLL $\approx$ -14.6 dB

   $$EF(47.1°) = \cos(47.1°) = 0.681$$

   $$20\log_{10}(0.681) = -3.3 \text{ dB}$$

   Pattern multiplication is a product, so in dB the two levels add:

   $$\text{SLL} \approx -11.3 \text{ dB} + (-3.3 \text{ dB}) = -14.6 \text{ dB}$$

   This is an estimate: the product peaks slightly closer to broadside, at $44.2°$, where it is -14.4 dB .

   (b) The array factor of this array has nulls at $\pm 30°$ and $\pm 90°$. Which nulls survive in the total pattern, and what is the total pattern at $\theta = 90°$?

   $|F(90°)|$ = 0

   All of them survive: a zero in either factor is a zero in the product. At $\theta = 90°$ both factors vanish, since $\cos(90°) = 0$ and the $m = 2$ array-factor null also sits there, so the total is zero (a deeper, wider null than either factor produces alone).

   (c) The element factor changed the sidelobe by 3.3 dB but changed the half-power beamwidth by almost nothing. Explain why, in terms of how $\cos(\theta)$ varies near broadside.

   $\cos(\theta)$ is flat near $\theta = 0$: at the $\pm 13.2°$ half-power edges of the four-element beam it is still 0.974, or -0.23 dB , so it scales the peak and the half-power points by nearly the same amount and the width only narrows from $26.3°$ to $25.4°$. Out at $47°$ the cosine has fallen by 3.3 dB , which is why the sidelobe moved. A directive element reshapes the skirts of a pattern; the array factor sets the main lobe.

   (d) A colleague proposes fixing a beam that is too wide by replacing the short dipoles with more directive elements, keeping N and d the same. Say whether that works and what it would change instead.

   It does not fix the beamwidth. The main lobe is set by the array factor, whose width follows the total length Nd; a more directive element is nearly constant across that narrow main lobe and only scales it. What the change does deliver is lower sidelobes and less wide-angle radiation. To narrow the beam you must increase Nd, by adding elements or increasing the spacing within the grating-lobe limit.

4. **Spacing, the visible region, and grating lobes.** A uniform linear array is steered to θ_0 by a progressive phase, so

$$\psi = kd(\sin(\theta) - \sin(\theta_0))$$

and grating lobes appear wherever

$$\sin(\theta_g) = \sin(\theta_0) \pm \frac{m\lambda}{d}$$

(a) An unsteered array uses $d = \lambda$. Find every grating lobe in the visible region, and give the width of the visible region in ψ.

θ_g = $\pm 90^\circ$

ψ width = 720°

$$\sin(\theta_g) = 0 \pm \frac{\lambda}{\lambda} = \pm 1 \quad \Longrightarrow \quad \theta_g = \pm 90^\circ$$

The visible region spans $2kd = 4\pi(d/\lambda) = 4\pi$, which is two full periods of the array factor, so exactly one repeat of the main lobe reaches each edge of real space.

(b) A second array uses $d = 0.75\lambda$ and is steered to $\theta_0 = 30^\circ$. Apply the avoidance criterion, then find any grating lobe that is present.

$d_{\max}$ = 0.667λ

θ_g = -56.4°

$$d_{\max} = \frac{\lambda}{1 + |\sin(30^\circ)|} = \frac{\lambda}{1.5} = 0.667\lambda$$

$0.75\lambda > 0.667\lambda$, so the spacing violates the criterion and a grating lobe is in view.

$$\sin(\theta_g) = 0.5 - \frac{1}{0.75} = 0.5 - 1.333 = -0.833$$

$$\theta_g = -56.4^\circ$$

The $+$ branch gives 1.833, which is not a real angle.

(c) You must scan to $\pm 45^\circ$ at 10.3 GHz with no grating lobe. Find the largest usable element spacing in millimeters, and state the trade-off that stops you from simply making d much smaller.

$d_{\max}$ = 17.1 mm

$$d < \frac{\lambda}{1 + \sin(45^\circ)} = \frac{29.1 \text{ mm}}{1.707} = 17.1 \text{ mm}$$

Smaller spacing is safe from grating lobes but shortens the array for a given element count, so the beam widens as $0.886\lambda/Nd$; holding the beamwidth then requires more elements, more phase shifters, and more channels. Close spacing also increases mutual coupling between elements. The usual compromise is 0.4λ to 0.5λ.

(d) In the lab you will turn on every third element of the PHASER, making the effective spacing 42 mm at 10.3 GHz. Predict the grating lobe angles for the unsteered array.

θ_g = $\pm 43.9^\circ$

$$\frac{d}{\lambda} = \frac{42}{29.1} = 1.44$$

$$\sin(\theta_g) = \pm\frac{1}{1.44} = \pm 0.693 \quad \Longrightarrow \quad \theta_g = \pm 43.9^\circ$$

$m = 2$ would need $\sin(\theta) = 1.39$, so only the first pair appears. Expect three lobes of equal height: the main lobe at 0° and grating lobes near $\pm 44^\circ$.

5. **The array as a sampled aperture.** Two unsteered uniform arrays are candidates for the same job. Array A has $N = 8$ elements at $d = \lambda/2$. Array B has $N = 4$ elements at $d = \lambda$.

   (a) Find the total length and the half-power beamwidth of each array.

   Nd (both) = 4λ

   A : $Nd = 8(0.5\lambda) = 4\lambda$

   B : $Nd = 4(1.0\lambda) = 4\lambda$

   HPBW (both) = 12.7°

   $$\text{HPBW} \approx \frac{0.886\lambda}{4\lambda} = 0.222 \text{ rad} = 12.7^\circ$$

   Both arrays are the same length, so both have the same main lobe.

   (b) Find the first sidelobe level and the grating lobe angles of each array.

   Both are uniformly excited, so both show the uniform first sidelobe near -13 dB (-12.8 dB for A at $N = 8$, -11.3 dB for B at $N = 4$).

   A: θ_g = none

   A : $\sin(\theta_g) = \pm\dfrac{\lambda}{0.5\lambda} = \pm 2$ (not real, none)

   B: θ_g = $\pm 90^\circ$

   B : $\sin(\theta_g) = \pm\dfrac{\lambda}{\lambda} = \pm 1 \quad \Longrightarrow \quad \pm 90^\circ$

   (c) Array B uses half the elements for the same beamwidth. State which array you would build and why, and say what happens to Array B's pattern the moment we steer it off broadside.

   Build Array A. Both have the same main lobe because both have the same length Nd, but Array B radiates a full-strength grating lobe at $\pm 90^\circ$ at broadside, and steering slides that lobe into the forward half-space while the intended beam moves the other way — so power and receive sensitivity go away from the commanded direction. Using four fewer channels leaves the array unable to steer at all. At a fixed length, the element count sets the spacing and with it the grating lobes; the length sets the beamwidth.

Documentation: _____