Learning Objective 1.6: I can set up and interpret the radiation integrals to derive the far-field pattern of a current distribution.

Show your work. Write a **Documentation** line at the end (**None** if you did not collaborate).

1. **Setting up the integral.** Answer each in one or two sentences (or one equation).

   (a) Why do we solve for the magnetic vector potential $\vec{A}$ instead of attacking $\vec{E}$ directly? What property of $\vec{B}$ guarantees $\vec{A}$ exists?

   $\vec{E}$ and $\vec{H}$ are coupled and the source enters through a curl. $\vec{A}$ absorbs the source into a single equation ($\nabla^2\vec{A} + k^2\vec{A} = -\mu\vec{J}$) whose solution is already known. It exists because $\nabla \cdot \vec{B} = 0$ always, and a divergence-free field is always the curl of something.

   (b) The far field uses *two different* approximations to $R = |\vec{r} - \vec{r}'|$. Write both and state where you use each.

   Amplitude: $1/R \approx 1/r$ (a fractional error of order r'/r). Phase: $R \approx r - \hat{\mathbf{r}} \cdot \vec{r}'$, because $k\hat{\mathbf{r}} \cdot \vec{r}'$ is several radians and controls whether contributions add or cancel.

   (c) Write the far-field vector potential in the form (spherical wave) $\times$ (radiation vector), and say what each factor depends on.

$$\vec{A} = \frac{\mu e^{-jkr}}{4\pi r}\vec{N}(\theta,\phi), \qquad \vec{N} = \int_{V'} \vec{J}(\vec{r}')e^{+jk\hat{\mathbf{r}}\cdot\vec{r}'}\,dV'$$

   e^{-jkr}/r depends only on *distance* and is the same for every antenna; $\vec{N}$ depends only on *direction* and carries everything that makes this antenna different.

   (d) A $\hat{\mathbf{z}}$-directed current gives $N_\theta = -N_z \sin(\theta)$. Where does that $\sin(\theta)$ come from, and what does it predict about radiation along the wire axis?

   It is the *projection* of a $\hat{\mathbf{z}}$-directed current onto $\hat{\boldsymbol{\theta}}$ — not part of the integral. At $\theta = 0^\circ$ the projection vanishes, so a wire antenna radiates nothing off its own ends.

2. **The term we throw away.** A reflector antenna has largest dimension $D = 0.8$m and operates at 6GHz. The quadratic term dropped from the expansion of R leaves a worst-case phase error $\Delta\phi = \pi D^2/(4\lambda r)$.

   (a) Find λ.

   $$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{6 \times 10^9} = 0.05\text{m}$$

   λ = 0.05m

   (b) A test range puts the source at $r = 20$m. Find $\Delta\phi$ in degrees.

   $$\Delta\phi = \frac{\pi(0.8)^2}{4(0.05)(20)} = \frac{2.011}{4} = 0.503 \text{ rad} \approx 28.8^\circ$$

   $\Delta\phi$ = $\approx 28.8^\circ$

   (c) Find the smallest r for which $\Delta\phi \leqslant 22.5^\circ$ ($\pi/8$). Compare it to $2D^2/\lambda$.

   $$\frac{\pi D^2}{4\lambda r} \leqslant \frac{\pi}{8} \quad \longrightarrow \quad r \geqslant \frac{2D^2}{\lambda} = \frac{2(0.8)^2}{0.05} = 25.6\text{m}$$

   r_{min} = 25.6m

   They are the *same* criterion — the far-field distance is exactly the distance at which the parallel-ray approximation becomes accurate.

   (d) The range in part (b) is too short. Is the measured *pattern* wrong because the amplitude approximation failed, the phase approximation failed, or both?

   The **phase** approximation. The $1/R \approx 1/r$ amplitude approximation is still excellent at 20m; it is the leftover 28.8° of quadratic phase across the aperture that distorts the pattern (fills nulls, raises sidelobes).

3. **Uniform line source.** A $\hat{z}$-directed filament carries a uniform current I_0 over $-L/2 \leqslant z' \leqslant L/2$, with $L = 3\lambda$. Recall the space factor $S(\theta) = |\sin(u)/u|$ with $u = \frac{kL}{2}\cos(\theta)$.

   (a) Evaluate the radiation integral to show where $S(\theta)$ comes from.

$$N_z(\theta) = I_0 \int_{-L/2}^{L/2} e^{+jkz'\cos(\theta)} dz'$$
$$= I_0 \frac{2\sin\left(\frac{kL}{2}\cos(\theta)\right)}{k\cos(\theta)}$$
$$= I_0 L \frac{\sin(u)}{u}, \quad u = \frac{kL}{2}\cos(\theta)$$

N_z = $I_0 L \sin(u)/u$

   (b) Find the angles of the first nulls.

First null at $u = \pi$, i.e. $\cos(\theta) = \lambda/L = 1/3$:

$$\theta = 70.5^\circ \quad \text{and} \quad 109.5^\circ$$

θ_{null} = $70.5^\circ,\ 109.5^\circ$

   (c) Estimate the half-power beamwidth with $0.886\lambda/L$, then check it against the exact half-power condition $u = 1.392$.

Rule of thumb: $0.886/3 = 0.295$ rad $\approx 16.9^\circ$. Exact: $\cos(\theta) = 1.392/(3\pi) = 0.1477 \rightarrow \theta = 81.5^\circ$, so HPBW $= 2(90^\circ - 81.5^\circ) = 17.0^\circ$. The rule of thumb is good to 0.1° here.

rule of thumb = 16.9°

exact = 17.0°

   (d) The design is changed to $L = 6\lambda$. What happens to the beamwidth, and what happens to the first sidelobe level? Justify both from the Fourier relationship.

Beamwidth **halves** to $\approx 8.5^\circ$ — stretching the current distribution squeezes its transform. The first sidelobe **does not move**: it stays at -13.3 dB , because the *shape* sets it (the abrupt edges of a uniform distribution), not the length. Beamwidth is bought with size; sidelobes are bought with taper.

4. **Steering with a phase slope.** A uniform line source of length $L = 4\lambda$ carries a progressive phase that moves the pattern peak off broadside to $\theta_0 = 60^\circ$. The excitation is $I(z') = I_0 e^{-jkz'\cos(\theta_0)}$.

   (a) Show from the radiation integral that the peak lands at θ_0, and state the peak angle for this design.

   $$N_z(\theta) = I_0 \int e^{+jkz'(\cos(\theta)-\cos(\theta_0))} dz'$$

   The integrand is in phase across the whole aperture — and the sum therefore maximum — when $\cos(\theta) = \cos(\theta_0)$, i.e. $\theta = \theta_0 = 60^\circ$. A linear phase across the source shifts the transform; it does not reshape it.

   θ_{peak} = $\theta_0 = 60^\circ$

   (b) Find the total phase difference (in degrees) between the two ends of the aperture.

   $$\Delta\phi = kL\cos(\theta_0) = 2\pi(4)(0.5) = 4\pi \text{ rad} = 720^\circ$$

   $\Delta\phi$ = $720^\circ\,(4\pi)$

   (c) The steered beamwidth is approximately $0.886\lambda/(L\sin(\theta_0))$. Evaluate it and compare with the broadside value.

   Broadside: $0.886/4 = 0.2215$ rad $= 12.7^\circ$. Steered: $12.7^\circ/\sin(60^\circ) = 14.7^\circ$. The beam *broadens* because the aperture looks shorter when viewed from 60° — its projected length is $L\sin(\theta_0)$.

   HPBW = $\approx 14.7^\circ$

5. **The half-wave dipole.** The sinusoidal current $I(z') = I_0 \sin\left(k\left(\frac{L}{2} - |z'|\right)\right)$ with $L = \lambda/2$ produces the far-field pattern

$$|F(\theta)| = \left| \frac{\cos\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)} \right|$$

(a) Evaluate $|F|$ at $\theta = 90°$ and at $\theta = 0°$, and interpret each.

At $\theta = 90°$: $\cos(\theta) = 0$, so $|F| = \cos(0)/1 = 1$ — maximum, broadside to the wire. At $\theta = 0°$: numerator $\cos(\pi/2) = 0$ and denominator $\to 0$; the limit is 0 — a genuine null off the ends of the wire.

$|F(90°)|$ = 1 (peak)

$|F(0°)|$ = 0 (null)

(b) Evaluate $|F|$ at $\theta = 45°$, in dB relative to the peak.

$$|F| = \frac{\cos\left(\frac{\pi}{2}(0.7071)\right)}{0.7071} = \frac{\cos(63.6°)}{0.7071}$$
$$= \frac{0.4440}{0.7071} = 0.628$$
$$\to 20\log_{10}(0.628) \approx -4.0 \text{ dB}$$

$|F(45°)|$ = 0.628 (−4.0 dB)

(c) An infinitesimal dipole has $|F| = \sin(\theta)$, which at $45°$ is 0.707 (-3.0 dB). The half-wave dipole is 1 dB further down at the same angle. Explain physically why the half-wave pattern is *sharper*, in terms of the current distribution.

The half-wave dipole's current is *tapered* — peaked at the feed and zero at the tips — so the radiating current is concentrated near the middle of a structure that is a half wavelength long, rather than spread uniformly over a vanishingly short one. A longer effective distribution has a narrower transform, so the beam is narrower: HPBW $= 78.1°$ versus $90°$, and $D = 1.64$ (2.15 dBi) versus 1.5 (1.76 dBi).

(d) The radiation integral is exact once $\vec{J}$ is known. State what makes $\vec{J}$ hard to know, and name the numerical method that solves for it.

The fields set the current, and the current sets the fields — a self-consistent (boundary-value) problem. The **method of moments** discretizes the structure, enforces the boundary condition on each segment, and solves a linear system for the segment currents; it then runs this same radiation integral to get the pattern. (This is what NEC does.)

6. **Reading the Fourier relationship.** Four aperture distributions of the *same* length L produce first sidelobes of -13.3 dB , -23 dB , -26.5 dB , and -31.5 dB . Work these parts from the **taper table in Lesson 6**, which pairs each distribution with its first-sidelobe level and also records the price: relative to uniform illumination, a taper broadens the main beam by roughly $1.3\times$ to $1.6\times$.

   (a) Match each level to its distribution: uniform, cosine taper, cosine-squared taper, triangular.

   $$\text{uniform} \to -13.3\text{ dB}, \qquad \text{cosine} \to -23\text{ dB}$$
   $$\text{triangular} \to -26.5\text{ dB}, \qquad \text{cosine-squared} \to -31.5\text{ dB}$$

   The smoother the distribution goes to zero at the edges, the less high-space-frequency content it has, and the lower the sidelobes.

   (b) All four have the same L. Rank their half-power beamwidths, put an approximate broadening factor on each, and explain the trade.

   Uniform is *narrowest* ($1.00\times$, by definition), then cosine ($\approx 1.34\times$), then triangular ($\approx 1.43\times$), then cosine-squared (widest, $\approx 1.63\times$). The ranking is the sidelobe ranking read backwards: tapering suppresses the outer part of the aperture, so the *effective* length shrinks and the beam broadens. Same L does **not** mean same beamwidth. **Sidelobe level is bought with beamwidth** – the central trade of aperture design (L15, L22).

   (c) Your requirement is a 2° beam with sidelobes below -25 dB . Which knob sets which requirement? Size the aperture, in wavelengths, both before and after you apply the taper you chose.

   L (uniform) = 25.4λ

   L (tapered) = 36.7λ

   Length sets the beamwidth, **taper shape** sets the sidelobe level. Start with uniform illumination and $\theta_{HP} \approx 0.886\lambda/L$:

   $$\theta_{HP} = 2^\circ = 2\left(\frac{\pi}{180}\right) = 0.0349 \text{ rad}$$
   $$L = \frac{0.886\lambda}{0.0349} \approx 25.4\lambda$$

   Uniform gives only -13.3 dB sidelobes, so pick the cheapest taper that clears -25 dB : **triangular**, at -26.5 dB . Its beamwidth constant is $1.28\lambda/L$ ($\approx 1.44\times$ uniform), so grow the aperture by the same factor to hold 2°:

   $$L = \frac{1.28\lambda}{0.0349} \approx 1.44\,(25.4\lambda) \approx 36.7\lambda$$

   Cosine-squared would also clear the spec but costs $1.63\times$, i.e. $\approx 41\lambda$ – more aperture than the requirement is asking you to buy.

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