Learning Objective 5.4: I can apply the electric field boundary conditions to calculate electric fields on both sides of a material boundary.

Show your work. Write a **Documentation** line at the end (**None** if you did not collaborate). Ignore fringing in every parallel-plate problem.

1. **Parallel plates.** Two plates of area A are a distance d apart, with a dielectric of permittivity ϵ between them. The plate at $z = d$ carries $+\rho_s$ and the plate at $z = 0$ carries $-\rho_s$.

   (a) Use the conductor boundary condition to find **E** between the plates, then integrate it to derive $C = \epsilon A/d$.

   At the top plate the outward normal into the gap is $-\hat{\mathbf{z}}$:

   $$\mathbf{D}\cdot(-\hat{\mathbf{z}}) = \rho_s \implies \mathbf{E} = -\hat{\mathbf{z}}\frac{\rho_s}{\epsilon}$$

   $$V = -\int_0^d \left(-\hat{\mathbf{z}}\frac{\rho_s}{\epsilon}\right)\cdot\hat{\mathbf{z}}\,dz = \frac{\rho_s d}{\epsilon}$$

   $$C = \frac{Q}{V} = \frac{\rho_s A}{\rho_s d/\epsilon} = \frac{\epsilon A}{d}$$

   (b) The plates measure 2 cm $\times$ 3 cm and are 0.5 mm apart, filled with a film of $\epsilon_r = 3.2$. Find C.

   $$C = \frac{3.2 \times 8.854 \times 10^{-12} \times 6 \times 10^{-4}}{5 \times 10^{-4}} = 34.0\ \text{pF}$$

   C = 34.0 pF

   (c) The capacitor is charged to 100 V. Find Q, ρ_s, and $|\mathbf{E}|$, and check that $|\mathbf{E}| = \rho_s/\epsilon$.

   $$Q = CV = 3.40\ \text{nC}$$

   $$\rho_s = \frac{Q}{A} = 5.67\ \mu\text{C/m}^2$$

   $$|\mathbf{E}| = \frac{V}{d} = 200\ \text{kV/m}$$

   $$\frac{\rho_s}{\epsilon} = \frac{5.67 \times 10^{-6}}{2.83 \times 10^{-11}} = 200\ \text{kV/m}$$

   Q = 3.40 nC

   ρ_s = 5.67 μC/m^2

   $|\mathbf{E}|$ = 200 kV/m

2. **Coaxial capacitance.** A coaxial line has inner radius a, shield radius b, and a dielectric of permittivity ϵ.

(a) Place $+\rho_\ell$ per meter on the inner conductor. Use Gauss's law on a cylinder of radius ρ and length ℓ to find **E** for $a < \rho < b$, then derive $C' = 2\pi\epsilon/\ln(b/a)$.

The field is radial and depends only on ρ; the end caps carry no flux.

$$D_\rho\,(2\pi\rho\ell) = \rho_\ell\,\ell \implies \mathbf{E} = \hat{\boldsymbol{\rho}}\frac{\rho_\ell}{2\pi\epsilon\rho}$$

$$V = -\int_b^a \frac{\rho_\ell}{2\pi\epsilon\rho}\,d\rho = \frac{\rho_\ell}{2\pi\epsilon}\ln\frac{b}{a}$$

$$C' = \frac{Q/\ell}{V} = \frac{\rho_\ell}{\frac{\rho_\ell}{2\pi\epsilon}\ln(b/a)} = \frac{2\pi\epsilon}{\ln(b/a)}$$

(b) RG-59 has $a = 0.29$ mm, $b = 1.85$ mm, and solid polyethylene, $\epsilon_r = 2.25$. Find C' and the capacitance of a 15 m run.

$$C' = \frac{2\pi \times 2.25 \times 8.854 \times 10^{-12}}{\ln(1.85/0.29)}$$

$$= \frac{1.252 \times 10^{-10}}{1.853} = 67.5 \text{ pF/m}$$

$$C = 67.5 \times 15 = 1.01 \text{ nF}$$

C' = 67.5 pF/m

C = 1.01 nF

(c) Module 3 shows that $L' = (\mu_0/2\pi)\ln(b/a)$. Use it to find Z_0 for RG-59, and explain why the cable is sold as 75 Ω line.

$$L' = 2 \times 10^{-7} \times 1.853 = 371 \text{ nH/m}$$

$$Z_0 = \sqrt{\frac{L'}{C'}} = \sqrt{\frac{3.71 \times 10^{-7}}{6.75 \times 10^{-11}}} = 74.1\ \Omega$$

That is within 2% of 75 Ω.

Z_0 = 74.1 Ω

3. **Two layers.** Plates of area 0.01 m^2 hold two stacked layers: 1 mm with $\epsilon_{r1} = 2$ and 1 mm with $\epsilon_{r2} = 8$. The capacitor is charged to 500 V.

   (a) Explain why **D** is the same in both layers.

   The field is normal to the interface, and the interface carries no free charge, so the normal boundary condition makes $D_{n1} = D_{n2}$.

   (b) Find C, and check it as two capacitors in series.

$$C = \frac{A}{d_1/\epsilon_1 + d_2/\epsilon_2} = \frac{\epsilon_0 \times 0.01}{5 \times 10^{-4} + 1.25 \times 10^{-4}}$$
$$= 142 \text{ pF}$$
$$C_1 = \frac{2\epsilon_0 A}{10^{-3}} = 177 \text{ pF}$$
$$C_2 = \frac{8\epsilon_0 A}{10^{-3}} = 708 \text{ pF}$$
$$\frac{C_1 C_2}{C_1 + C_2} = \frac{177 \times 708}{885} = 142 \text{ pF}$$

C = 142 pF

   (c) Find the field and the voltage in each layer.

$$D = \frac{V}{d_1/\epsilon_1 + d_2/\epsilon_2} = \frac{500\,\epsilon_0}{6.25 \times 10^{-4}}$$
$$E_1 = \frac{D}{2\epsilon_0} = 400 \text{ kV/m}, \qquad V_1 = E_1 d_1 = 400 \text{ V}$$
$$E_2 = \frac{D}{8\epsilon_0} = 100 \text{ kV/m}, \qquad V_2 = E_2 d_2 = 100 \text{ V}$$

E_1 = 400 kV/m

E_2 = 100 kV/m

   (d) Both materials break down at 20 MV/m. Which layer fails first as the voltage rises, and at what total voltage?

   The $\epsilon_{r1} = 2$ layer, which carries four times the field of the other. Its field reaches 20 MV/m when the total voltage is $500 \times 20 \times 10^6/4 \times 10^5 = 25$ kV.

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