Learning Objective 5.4: I can apply the electric field boundary conditions to calculate electric fields on both sides of a material boundary.

Show your work. Write a **Documentation** line at the end (**None** if you did not collaborate). In every problem, $\hat{\mathbf{n}}$ points from region 2 into region 1.

1. **Deriving the conditions.** Two dielectrics meet at a flat boundary that carries a surface charge density ρ_s.

   (a) Apply Gauss's law to a pillbox with faces of area ΔA, one in each region, and height $h \to 0$. Derive $D_{n1} - D_{n2} = \rho_s$.

$$\oint \mathbf{D}\cdot d\mathbf{s} = \int_{\text{top}} \mathbf{D}_1\cdot\hat{\mathbf{n}}\, ds + \int_{\text{bottom}} \mathbf{D}_2\cdot(-\hat{\mathbf{n}})\, ds + \int_{\text{side}} \mathbf{D}\cdot d\mathbf{s}$$

$$= D_{n1}\,\Delta A - D_{n2}\,\Delta A + 0 = Q_{\text{enc}} = \rho_s\,\Delta A$$

$$D_{n1} - D_{n2} = \rho_s$$

   The side has area proportional to h, so its flux vanishes as $h \to 0$, and so does the contribution $\rho_v\,\Delta A\,h$ of any volume charge. ΔA is small enough that $\mathbf{D}$ and ρ_s are uniform over it.

   (b) With $\rho_s = 0$, let θ_1 and θ_2 be the angles between the field and the normal on each side. Show that $\tan\theta_1/\tan\theta_2 = \epsilon_1/\epsilon_2$.

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{E_{t1}/E_{n1}}{E_{t2}/E_{n2}} = \frac{E_{t1}}{E_{t2}}\cdot\frac{E_{n2}}{E_{n1}}$$

   The tangential condition makes $E_{t1}/E_{t2} = 1$. With $\rho_s = 0$ the normal condition is $\epsilon_1 E_{n1} = \epsilon_2 E_{n2}$, so $E_{n2}/E_{n1} = \epsilon_1/\epsilon_2$.

2. **Across a dielectric boundary.** The plane $z = 0$ separates region 1, $z > 0$, with $\epsilon_{r1} = 2$, from region 2, $z < 0$, with $\epsilon_{r2} = 5$. The boundary carries no free charge. Just above it, $\mathbf{E}_1 = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}5$ V/m.

   (a) Find $\mathbf{E}_2$ just below the boundary.

   The normal is $\hat{\mathbf{z}}$. Tangential: $E_{2x} = 2$, $E_{2y} = -3$. Normal:

   $$\epsilon_{r1} E_{1z} = \epsilon_{r2} E_{2z} \implies E_{2z} = \frac{2 \times 5}{5} = 2$$

   $\mathbf{E}_2 =$ $\hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}2$ V/m

   (b) Find $\mathbf{D}_2$.

   $$\mathbf{D}_2 = 5\epsilon_0 \mathbf{E}_2 = \hat{\mathbf{x}}88.5 - \hat{\mathbf{y}}133 + \hat{\mathbf{z}}88.5 \text{ pC/m}^2$$

   $\mathbf{D}_2 =$ $\epsilon_0 \left(\hat{\mathbf{x}}10 - \hat{\mathbf{y}}15 + \hat{\mathbf{z}}10\right)$

   (c) Find the angle each field makes with the normal, and check your answers against part (b) of Question 1.

   $$|E_t| = \sqrt{2^2 + 3^2} = 3.61 \text{ V/m}$$
   $$\theta_1 = \tan^{-1} \frac{3.61}{5} = 35.8^\circ$$
   $$\theta_2 = \tan^{-1} \frac{3.61}{2} = 61.0^\circ$$
   $$\frac{\tan\theta_1}{\tan\theta_2} = \frac{0.721}{1.80} = 0.400 = \frac{2}{5}$$

   $\theta_1 =$ 35.8°

   $\theta_2 =$ 61.0°

   (d) In which region is $|\mathbf{E}|$ larger, and why?

   Region 1: $|\mathbf{E}_1| = \sqrt{38} = 6.16$ V/m and $|\mathbf{E}_2| = \sqrt{17} = 4.12$ V/m. The tangential parts are equal, and the normal part is smaller in region 2, which has the larger permittivity.

3. **A charged boundary.** The plane $z = 0$ separates air, region 1 at $z > 0$, from a dielectric with $\epsilon_r = 4$, region 2 at $z < 0$. The boundary carries $\rho_s = 88.5$ nC/m^2. Just above it, $\mathbf{E}_1 = \hat{\mathbf{x}}10 + \hat{\mathbf{z}}30$ kV/m.

   (a) Find $\mathbf{E}_2$.

   Tangential: $E_{2x} = 10$ kV/m. Normal, with $\hat{\mathbf{n}} = \hat{\mathbf{z}}$:

   $$D_{2z} = D_{1z} - \rho_s = \epsilon_0 \left(3 \times 10^4\right) - 8.85 \times 10^{-8}$$
   $$= 2.66 \times 10^{-7} - 0.885 \times 10^{-7} = 1.77 \times 10^{-7} \text{ C/m}^2$$
   $$E_{2z} = \frac{D_{2z}}{4\epsilon_0} = \frac{1.77 \times 10^{-7}}{3.54 \times 10^{-11}} = 5 \text{ kV/m}$$

   $\mathbf{E}_2 =$ $\hat{\mathbf{x}}10 + \hat{\mathbf{z}}5$ kV/m

   (b) Without the surface charge, what would E_{2z} be? Explain the difference in one sentence.

   $$E_{2z} = \frac{\epsilon_0 \times 30}{4\epsilon_0} = 7.5 \text{ kV/m}$$

   The positive surface charge supplies some of the normal flux above the boundary, so less of it has to arrive from below.

4. **At a conductor.** A conductor fills $z < 0$. Above it is a dielectric with $\epsilon_r = 2.5$, and the field just above the surface is $\mathbf{E} = \hat{\mathbf{z}}40$ kV/m.

   (a) Find the surface charge density on the conductor.

   $$\rho_s = D_n = \epsilon_r \epsilon_0 E = 2.5 \times 8.854 \times 10^{-12} \times 4 \times 10^4$$
   $$= 0.885 \ \mu\text{C/m}^2$$

   $\rho_s =$ $0.885 \ \mu$C/m^2

   (b) A classmate reports a field of $\hat{\mathbf{x}}5 + \hat{\mathbf{z}}40$ kV/m just above the same surface. Explain why the measurement cannot be right.

   The field inside the conductor is zero, so the tangential condition requires the tangential field just outside to be zero too. A 5 kV/m x component would drive charge along the surface, and the charge would move until it vanished.

   (c) What is the volume charge density inside the conductor? Justify your answer with Gauss's law.

   Zero. Inside, $\mathbf{D} = 0$, so $\rho_v = \nabla \cdot \mathbf{D} = 0$; the conductor's charge sits entirely on its surface.

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