Learning Objective 5.3: I can calculate the electric field from the scalar electric potential in a variety of geometries.

Show your work. Take $\epsilon = \epsilon_0 = 8.854 \times 10^{-12}$ F/m unless told otherwise. Write a **Documentation** line at the end (**None** if you did not collaborate).

1. **Poisson's and Laplace's equations.**

   (a) Starting from $\nabla \cdot \mathbf{D} = \rho_v$, derive Poisson's equation for a homogeneous material.

   $$\nabla \cdot (\epsilon \mathbf{E}) = \rho_v \;\Rightarrow\; \epsilon \nabla \cdot (-\nabla V) = \rho_v \;\Rightarrow\; \nabla^2 V = -\frac{\rho_v}{\epsilon}$$

   ϵ leaves the divergence because it is the same everywhere.

   (b) Which of these potentials can exist in a charge-free region? For any that cannot, find ρ_v in free space.

   (1) $V_1 = 3x^2 - 3y^2 + 2z$

   (2) $V_2 = x^2 + y^2 + z^2$

   (3) $V_3 = \ln \rho$ (cylindrical, $\rho > 0$)

   (4) $V_4 = 1/r$ (spherical, $r > 0$)

   (1) $6 - 6 + 0 = 0$: yes. (2) $2 + 2 + 2 = 6$: no; $\rho_v = -6\epsilon_0 = -5.31 \times 10^{-11}$ C/m^3. (3) $\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \cdot \frac{1}{\rho}\right) = 0$: yes. (4) $\frac{1}{r^2}\frac{d}{dr}\left(r^2 \cdot \left(-\frac{1}{r^2}\right)\right) = 0$: yes.

2. **Parallel plates.** Two large plates sit at $z = 0$ and $z = d$ in air.

(a) The lower plate is grounded and the upper plate is at 12 V, with $d = 2$ mm. Solve Laplace's equation for $V(z)$, and find V at $z = 0.5$ mm and **E**.

$d^2V/dz^2 = 0$ gives $V = Az + B$; $V(0) = 0$ gives $B = 0$, and $V(d) = 12$ gives $A = 12/d$.

$$V(z) = 12\,\frac{z}{d}, \qquad V(0.5\text{ mm}) = 3\text{ V}$$

$$\mathbf{E} = -\hat{\mathbf{z}}\,\frac{12}{2\times 10^{-3}} = -\hat{\mathbf{z}}\,6\text{ kV/m}$$

V = 3 V

E = $-\hat{\mathbf{z}}\,6$ kV/m

(b) Now both plates are grounded and the space between them holds a uniform charge density ρ_v. Solve Poisson's equation for $V(z)$ and find **E**.

$$\frac{d^2V}{dz^2} = -\frac{\rho_v}{\epsilon} \;\Rightarrow\; V = -\frac{\rho_v z^2}{2\epsilon} + Az + B$$

$$V(0) = 0:\ B = 0, \qquad V(d) = 0:\ A = \frac{\rho_v d}{2\epsilon}$$

$$V(z) = \frac{\rho_v}{2\epsilon}\,z\,(d - z), \qquad \mathbf{E} = -\hat{\mathbf{z}}\,\frac{dV}{dz} = -\hat{\mathbf{z}}\,\frac{\rho_v}{2\epsilon}\,(d - 2z)$$

(c) For $\rho_v = 1\,\mu$C/m^3 and $d = 1$ cm, find the largest potential between the plates and the field at the lower plate.

$$V\left(\tfrac{d}{2}\right) = \frac{\rho_v d^2}{8\epsilon_0} = \frac{10^{-6}\times 10^{-4}}{8\times 8.854\times 10^{-12}} = 1.41\text{ V}$$

$$\mathbf{E}(0) = -\hat{\mathbf{z}}\,\frac{\rho_v d}{2\epsilon_0} = -\hat{\mathbf{z}}\,565\text{ V/m}$$

$V_{\max}$ = 1.41 V

E(0) = $-\hat{\mathbf{z}}\,565$ V/m

3. **The coaxial line.** An air-filled coax has inner radius a and outer radius b.

(a) The inner conductor is grounded and the outer conductor is at V_0. Solve Laplace's equation for $V(\rho)$ and find **E**.

$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0$ gives $V = A\ln\rho + B$. $V(a) = 0$ gives $V = A\ln(\rho/a)$, and $V(b) = V_0$ gives $A = V_0/\ln(b/a)$:

$$V(\rho) = V_0\,\frac{\ln(\rho/a)}{\ln(b/a)}, \qquad \mathbf{E} = -\hat{\boldsymbol{\rho}}\,\frac{dV}{d\rho} = -\hat{\boldsymbol{\rho}}\,\frac{V_0}{\rho\ln(b/a)}$$

The field points inward, from the outer conductor at high potential.

(b) For $a = 1$ mm, $b = 4$ mm, and $V_0 = 100$ V, find V at $\rho = 2$ mm and $|\mathbf{E}|$ at $\rho = a$.

$$V = 100\,\frac{\ln 2}{\ln 4} = 50\text{ V}$$

$$|\mathbf{E}(a)| = \frac{100}{10^{-3}\times 1.386} = 72.1\text{ kV/m}$$

V = 50 V

$|\mathbf{E}(a)|$ = 72.1 kV/m

(c) An air coax with $b = 5$ mm must hold 1 kV between its conductors. Find the inner radius a that makes the largest field, $V_0/(a\ln(b/a))$, as small as possible, and that field. Air breaks down near 3 MV/m; is the design safe?

Minimize $V_0/f(a)$ by maximizing $f(a) = a\ln(b/a)$:

$$\frac{df}{da} = \ln\frac{b}{a} - 1 = 0 \;\Rightarrow\; a = \frac{b}{e} = 1.84\text{ mm}$$

$$|\mathbf{E}|_{\max} = \frac{V_0}{(b/e)\times 1} = \frac{e\,V_0}{b} = \frac{2.718\times 1000}{0.005} = 544\text{ kV/m}$$

That is well below 3 MV/m, so the design is safe.

a = 1.84 mm

$|\mathbf{E}|_{\max}$ = 544 kV/m

4. **Concentric spheres.** A metal sphere of radius $a = 1$ cm at $V_0 = 50$ V sits at the center of a grounded spherical shell of radius $b = 2$ cm, with air between them.

   (a) Using $\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right)$ for a potential that depends on r alone, solve Laplace's equation for $V(r)$ and find **E**.

   $r^2\, dV/dr = A$, so $dV/dr = A/r^2$ and $V = -A/r + B$. $V(b) = 0$ gives $B = A/b$, so $V = -A\,(1/r - 1/b)$. $V(a) = V_0$ fixes A:

   $$V(r) = V_0 \frac{1/r - 1/b}{1/a - 1/b}, \qquad \mathbf{E} = -\hat{\mathbf{r}} \frac{dV}{dr} = \hat{\mathbf{r}} \frac{V_0}{r^2\,(1/a - 1/b)}$$

   (b) Find V at $r = 1.5$ cm and $|\mathbf{E}|$ at the surface of the inner sphere.

   $1/a - 1/b = 100 - 50 = 50\ \text{m}^{-1}$:

   $$V = 50\,\frac{66.7 - 50}{50} = 16.7\ \text{V}$$
   $$|\mathbf{E}(a)| = \frac{50}{10^{-4} \times 50} = 10\ \text{kV/m}$$

   V = 16.7 V

   $|\mathbf{E}(a)|$ = 10 kV/m

5. **Solving by averaging.** A long square trough is divided by a grid into four interior points, two upper (T) and two lower (B). The top wall is at 100 V and the other three walls are grounded. Each interior point is the average of its four neighbors.

   (a) By symmetry the two upper points are equal, and so are the two lower points. Write the averaging rule for one upper and one lower point, and solve for T and B.

   An upper point has neighbors 100, 0 (side wall), T, and B; a lower point has T, 0, B, and 0:

   $$4T = 100 + T + B, \qquad 4B = T + B$$
   $$B = T/3, \qquad 3T = 100 + T/3$$
   $$T = 37.5\ \text{V}, \qquad B = 12.5\ \text{V}$$

   T = 37.5 V

   B = 12.5 V

Documentation: _____