Learning Objective 5.3: I can calculate the electric field from the scalar electric potential in a variety of geometries.

Show your work. Take $\epsilon = \epsilon_0 = 8.854 \times 10^{-12}$ F/m unless told otherwise. Write a **Documentation** line at the end (**None** if you did not collaborate).

1. **Two point charges.** In free space, $Q_1 = 4$ nC sits at the origin and $Q_2 = -2$ nC sits at $(3, 0, 0)$ m.

   (a) Starting from $V_B - V_A = -\int_A^B \mathbf{E} \cdot d\mathbf{l}$ and the field of a point charge, derive $V(r) = Q/(4\pi\epsilon r)$ with the reference $V = 0$ at infinity.

   Only the radial part of $d\mathbf{l}$ is parallel to $\mathbf{E} = \hat{\mathbf{r}} Q/(4\pi\epsilon r^2)$, so $\mathbf{E} \cdot d\mathbf{l} = Q\,dr/(4\pi\epsilon r^2)$. With A at infinity:

   $$V(r) - 0 = -\int_\infty^r \frac{Q}{4\pi\epsilon r_1^2}\, dr_1 = -\frac{Q}{4\pi\epsilon}\left[-\frac{1}{r_1}\right]_\infty^r = \frac{Q}{4\pi\epsilon r}$$

   (b) Find the potential at $P = (0, 4, 0)$ m.

   $R_1 = 4$ m and $R_2 = \sqrt{3^2 + 4^2} = 5$ m:

   $$V_P = 8.988 \times 10^9 \left(\frac{4 \times 10^{-9}}{4} - \frac{2 \times 10^{-9}}{5}\right)$$
   $$= 8.988\,(1 - 0.4) = 5.39 \text{ V}$$

   V_P = **5.39 V**

   (c) Find the potential at $S = (3, 4, 0)$ m, and the work to carry a 1 nC charge from P to S.

   $R_1 = 5$ m and $R_2 = 4$ m:

   $$V_S = 8.988\left(\frac{4}{5} - \frac{2}{4}\right) = 2.70 \text{ V}$$
   $$W = q\,(V_S - V_P) = 10^{-9}\,(2.70 - 5.39) = -2.70 \text{ nJ}$$

   V_S = **2.70 V**

   W = **−2.70 nJ**

   (d) A classmate says the work in part (c) depends on whether the charge is carried along a straight line or around a wide arc. Explain why it does not.

   The static field is conservative: the work from P to S equals $q(V_S - V_P)$, which depends only on the endpoints. Around any closed path $\oint \mathbf{E} \cdot d\mathbf{l} = 0$, so two paths between the same points cost the same work.

2. **The field from a potential.**

(a) Starting from $dV = -\mathbf{E} \cdot d\mathbf{l}$ and the definition of the gradient, $dV = \nabla V \cdot d\mathbf{l}$, show that $\mathbf{E} = -\nabla V$.

Both expressions give the same dV for every step $d\mathbf{l}$, so $(\mathbf{E} + \nabla V) \cdot d\mathbf{l} = 0$ for every direction of $d\mathbf{l}$. A vector with zero component in every direction is zero, so $\mathbf{E} = -\nabla V$.

(b) In rectangular coordinates, $V = 20xy^2 - 5z$ V, with lengths in meters. Find $\mathbf{E}$, and its value and magnitude at $(1, 1, 1)$ m.

$|\mathbf{E}|$ = 45.0 V/m

$$\begin{aligned}
\mathbf{E} &= -\left(\hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}\right) \\
&= -\hat{\mathbf{x}}\,20y^2 - \hat{\mathbf{y}}\,40xy + \hat{\mathbf{z}}\,5 \\
\mathbf{E}(1,1,1) &= -\hat{\mathbf{x}}\,20 - \hat{\mathbf{y}}\,40 + \hat{\mathbf{z}}\,5 \text{ V/m} \\
|\mathbf{E}| &= \sqrt{400 + 1600 + 25} = 45.0 \text{ V/m}
\end{aligned}$$

(c) In cylindrical coordinates, $V = 10\cos\phi/\rho$ V. Find $\mathbf{E}$, using $\nabla V = \hat{\boldsymbol{\rho}}\,\partial V/\partial\rho + \hat{\boldsymbol{\phi}}\,(1/\rho)\,\partial V/\partial\phi + \hat{\mathbf{z}}\,\partial V/\partial z$, and evaluate it at $\rho = 2$ m, $\phi = 0$.

$\mathbf{E}$ = $\hat{\boldsymbol{\rho}}\,2.5$ V/m

$$\begin{aligned}
\mathbf{E} &= -\hat{\boldsymbol{\rho}}\left(-\frac{10\cos\phi}{\rho^2}\right) - \hat{\boldsymbol{\phi}}\,\frac{1}{\rho}\left(-\frac{10\sin\phi}{\rho}\right) \\
&= \hat{\boldsymbol{\rho}}\,\frac{10\cos\phi}{\rho^2} + \hat{\boldsymbol{\phi}}\,\frac{10\sin\phi}{\rho^2} \\
\mathbf{E}(2,0) &= \hat{\boldsymbol{\rho}}\,2.5 \text{ V/m}
\end{aligned}$$

3. **A ring of charge.** A thin ring of radius a in the plane $z = 0$, centered on the z axis, carries a uniform line charge ρ_ℓ in free space.

(a) Derive the potential at a point $(0, 0, z)$ on the axis.

Every element $dQ = \rho_\ell \, a \, d\phi$ of the ring is the same distance $R = \sqrt{a^2 + z^2}$ from the point, so R comes out of the integral:

$$V(z) = \int_0^{2\pi} \frac{\rho_\ell \, a \, d\phi}{4\pi\epsilon\sqrt{a^2 + z^2}} = \frac{2\pi a \, \rho_\ell}{4\pi\epsilon\sqrt{a^2 + z^2}} = \frac{\rho_\ell \, a}{2\epsilon\sqrt{a^2 + z^2}}$$

(b) Find **E** on the axis from $-\nabla V$.

By symmetry the field on the axis has only a z component:

$$\mathbf{E} = -\hat{\mathbf{z}} \frac{dV}{dz} = -\hat{\mathbf{z}} \frac{\rho_\ell a}{2\epsilon} \left(-\frac{1}{2}\right) \frac{2z}{(a^2 + z^2)^{3/2}} = \hat{\mathbf{z}} \frac{\rho_\ell \, a \, z}{2\epsilon \, (a^2 + z^2)^{3/2}}$$

(c) For $a = 5$ cm and $\rho_\ell = 10$ nC/m, find V and $|\mathbf{E}|$ at $z = 5$ cm.

$$V = \frac{10^{-8} \times 0.05}{2 \times 8.854 \times 10^{-12} \times 0.0707} = 399 \text{ V}$$

$$|\mathbf{E}| = \frac{10^{-8} \times 0.05 \times 0.05}{2 \times 8.854 \times 10^{-12} \times (0.005)^{3/2}} = 3.99 \text{ kV/m}$$

V =	399 V
$\lvert\mathbf{E}\rvert$ =	3.99 kV/m

(d) Show that for $z \gg a$ the potential becomes that of a point charge equal to the ring's total charge. Then explain why the result of part (a) is not enough to find the field at a point off the axis.

For $z \gg a$, $\sqrt{a^2 + z^2} \approx z$, so $V \approx \rho_\ell a/(2\epsilon z) = (2\pi a \rho_\ell)/(4\pi\epsilon z) = Q/(4\pi\epsilon z)$. The gradient needs how V changes in every direction. Part (a) gives V only along the axis, so it says nothing about $\partial V/\partial\rho$ off the axis. On the axis, symmetry makes that component zero.

4. **A field that is not electrostatic.** A field is given as $\mathbf{E} = \hat{\mathbf{y}}\, x$ V/m, with x in meters.

   (a) Find $\oint \mathbf{E} \cdot d\mathbf{l}$ counterclockwise around the square with corners $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ m in the plane $z = 0$.

   $\oint$ = 1 V

   E has only a y component, so the two sides along x contribute nothing. Up the side at $x = 1$: $\int_0^1 1\, dy = 1$. Down the side at $x = 0$: $\mathbf{E} = 0$ there, so it contributes nothing.

   $$\oint \mathbf{E} \cdot d\mathbf{l} = 1 \text{ V}$$

   (b) Find $\nabla \times \mathbf{E}$, and state whether this field could be produced by static charges. Could it be written as $-\nabla V$?

   $$\nabla \times \mathbf{E} = \hat{\mathbf{z}} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = \hat{\mathbf{z}}\, 1 \text{ V/m}^2$$

   The curl is not zero and the loop integral is not zero, so the field is not conservative. Static charges cannot produce it, and no potential V has it as $-\nabla V$, because the curl of any gradient is zero.

Documentation: _____