Learning Objective 5.2: I can apply the integral and differential forms of Gauss's Law to calculate the electric field due to charge density in various geometries, including rectangular, cylindrical, and spherical coordinate systems.

Show your work. Write a **Documentation** line at the end (**None** if you did not collaborate).

1. **The point form in Cartesian coordinates.**

   (a) A small box has edges Δx, Δy, and Δz. Find the net flux of $\mathbf{D}$ out through its two faces normal to x, and show that dividing by the volume and letting the box shrink gives $\partial D_x/\partial x$.

   The outward normal is $+\hat{\mathbf{x}}$ on the face at $x+\Delta x$ and $-\hat{\mathbf{x}}$ on the face at x, each of area $\Delta y\,\Delta z$:

   $$\Psi_x = [D_x(x+\Delta x) - D_x(x)]\,\Delta y\,\Delta z$$
   $$\frac{\Psi_x}{\Delta x\,\Delta y\,\Delta z} = \frac{D_x(x+\Delta x) - D_x(x)}{\Delta x} \xrightarrow{\Delta x \to 0} \frac{\partial D_x}{\partial x}$$

   (b) The flux density in a region is $\mathbf{D} = \hat{\mathbf{x}}\,2xy + \hat{\mathbf{y}}\,x^2 + \hat{\mathbf{z}}\,3z^2$ nC/m^2, with x, y, z in meters. Find ρ_v at $(1, 2, 3)$.

   $\rho_v =$ 22 nC/m^3

   $$\rho_v = \nabla\cdot\mathbf{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial z}(3z^2)$$
   $$= 2y + 0 + 6z = 4 + 18 = 22 \text{ nC/m}^3$$

   (c) $D_y = x^2$ is not zero. Explain why it adds nothing to the divergence.

   D_y does not change with y, so the flux it carries into a small box through one y face leaves through the other. Only a component that changes along its own direction produces net flux.

2. **Cylindrical coordinates.** In the region $\rho \leqslant 2$ m, the flux density is $\mathbf{D} = \hat{\boldsymbol{\rho}}\,\rho^3/4$ nC/m^2, with ρ in meters.

   (a) Find ρ_v.

   Only the radial term of the cylindrical divergence survives:

   $$\rho_v = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \cdot \frac{\rho^3}{4}\right) = \frac{1}{\rho}\cdot \rho^3 = \rho^2 \text{ nC/m}^3$$

   ρ_v = $\boxed{\rho^2 \text{ nC/m}^3}$

   (b) Find the charge per meter of length inside $\rho = 2$ m by integrating ρ_v.

   $$\frac{Q}{L} = \int_0^{2\pi}\int_0^2 \rho^2\,\rho\,d\rho\,d\phi = 2\pi \cdot \frac{2^4}{4} = 8\pi = 25.1 \text{ nC/m}$$

   Q/L = $\boxed{8\pi = 25.1 \text{ nC/m}}$

   (c) Check your answer with the integral form of Gauss's law on a cylinder of radius 2 m and length L.

   At $\rho = 2$, $D_\rho = 2^3/4 = 2$ nC/m^2, and only the side of the cylinder carries flux:

   $$\frac{1}{L}\oint_S \mathbf{D}\cdot d\mathbf{s} = D_\rho \cdot 2\pi\rho = 2 \times 2\pi \times 2 = 8\pi \text{ nC/m}$$

   The two forms agree.

3. **Spherical coordinates.**

(a) Show that the flux density of a point charge, $\mathbf{D} = \hat{\mathbf{r}} Q/(4\pi r^2)$, has zero divergence for $r > 0$. Then explain how Gauss's law still gives a flux of Q through a sphere around the charge.

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{Q}{4\pi r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Q}{4\pi}\right) = 0$$

There is no charge anywhere except at the origin, where the formula fails. All of the charge sits at that one point, and any sphere around it encloses it, so the flux through the sphere is Q.

(b) A charge distribution produces $\mathbf{D} = \hat{\mathbf{r}} k/r$ for $r > 0$, where k is a constant. Find ρ_v.

$$\rho_v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{k}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(kr) = \frac{k}{r^2}$$

$\rho_v =$ $\boxed{k/r^2}$

(c) Find the charge inside a sphere of radius R two ways: by integrating ρ_v, and from the flux of $\mathbf{D}$ through the sphere.

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{k}{r^2} r^2 \sin\theta \, dr \, d\theta \, d\phi = kR \cdot 2 \cdot 2\pi = 4\pi kR$$

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{s} = \frac{k}{R} \cdot 4\pi R^2 = 4\pi kR$$

The two counts agree, as the divergence theorem requires.

Documentation: _____