Learning Objective 5.2: I can apply the integral and differential forms of Gauss's Law to calculate the electric field due to charge density in various geometries, including rectangular, cylindrical, and spherical coordinate systems.

Show your work. Write a **Documentation** line at the end (**None** if you did not collaborate).

1. **Flux and enclosed charge.** A point charge $Q = 6$ nC sits at the center of a closed cubical surface of side 2 m.

   (a) Find the net electric flux out of the cube.

   By Gauss's law the net flux equals the enclosed charge:

   $$\oint_S \mathbf{D}\cdot d\mathbf{s} = Q_{\text{encl}} = 6 \text{ nC}$$

   $\Psi =$ 6 nC

   (b) Find the flux through one face of the cube. Justify your answer from symmetry.

   The charge sits at the center, so the six faces are identical as seen from the charge, and each carries the same flux:

   $$\Psi_{\text{face}} = \frac{6 \text{ nC}}{6} = 1 \text{ nC}$$

   $\Psi_{\text{face}} =$ 1 nC

   (c) A second charge, -4 nC, is placed 3 m from the center, outside the cube. What is the net flux out of the cube now? Does the flux density **D** on the cube's surface change? Explain both answers.

   The net flux is still 6 nC: the new charge is outside, and every cone of flux from it enters the cube and leaves again. The flux density on the surface does change, since **D** is the sum of both charges' fields; the extra flux entering on one side exactly balances the extra flux leaving on the other.

   (d) Explain why Gauss's law, true for this cube, does not let us find **D** at a point on the cube's surface.

   On the cube, **D** is neither normal to the faces everywhere nor constant in magnitude: it is strongest at the center of each face and tilted near the edges. We cannot pull D out of the integral, so the law gives only the total flux.

2. **A uniformly charged ball.** A ball of radius $a = 5$ cm in free space carries a uniform charge density $\rho_v = 2\,\mu\text{C/m}^3$.

(a) Use symmetry to state the direction of **D** and the coordinates it depends on. Then derive **E** inside the ball, $r < a$, from Gauss's law.

The ball looks the same from every direction, so $\mathbf{D} = \hat{\mathbf{r}} D_r(r)$. On a sphere of radius $r < a$:

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = D_r r^2 \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi = 4\pi r^2 D_r$$

$$Q_{\text{encl}} = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho_v\, r_1^2 \sin\theta\, dr_1\, d\theta\, d\phi = \frac{4}{3}\pi r^3 \rho_v$$

$$4\pi r^2 D_r = \frac{4}{3}\pi r^3 \rho_v \quad\Longrightarrow\quad \mathbf{E} = \hat{\mathbf{r}}\,\frac{\rho_v r}{3\epsilon_0}$$

(b) Derive **E** outside the ball, $r > a$, and show that it is the field of a point charge equal to the ball's total charge.

The sphere now encloses the whole ball:

$$4\pi r^2 D_r = \frac{4}{3}\pi a^3 \rho_v = Q \quad\Longrightarrow\quad \mathbf{E} = \hat{\mathbf{r}}\,\frac{\rho_v a^3}{3\epsilon_0 r^2} = \hat{\mathbf{r}}\,\frac{Q}{4\pi\epsilon_0 r^2}$$

(c) Find E at $r = 2.5$ cm, 5 cm, and 10 cm.

$$E(a) = \frac{2\times 10^{-6}\times 0.05}{3\times 8.854\times 10^{-12}} = 3.76\text{ kV/m}$$

$$E(2.5\text{ cm}) = \tfrac{1}{2}E(a) = 1.88\text{ kV/m}$$

$$E(10\text{ cm}) = \left(\tfrac{5}{10}\right)^2 E(a) = 0.941\text{ kV/m}$$

$E(2.5\text{ cm})$ = 1.88 kV/m

$E(5\text{ cm})$ = 3.76 kV/m

$E(10\text{ cm})$ = 0.941 kV/m

3. **A coaxial line.** A coaxial line has an inner conductor of radius $a = 1$ mm, a shield of radius $b = 3.5$ mm, and a PTFE dielectric with $\epsilon_r = 2.1$. The inner conductor carries $\rho_l = 2$ nC/m and the shield carries $-\rho_l$.

   (a) Derive **E** in each of the three regions $\rho < a$, $a < \rho < b$, and $\rho > b$.

   By symmetry $\mathbf{D} = \hat{\boldsymbol{\rho}} D_\rho(\rho)$. On a closed cylinder of radius ρ and length L, the top and bottom contribute nothing, so

   $$\oint_S \mathbf{D}\cdot d\mathbf{s} = \int_0^L \int_0^{2\pi} D_\rho \, \rho \, d\phi \, dz = 2\pi\rho L D_\rho = Q_{\text{encl}}$$

   For $\rho < a$, the charge sits on the conductor's surface: $Q_{\text{encl}} = 0$, so $\mathbf{E} = 0$. For $a < \rho < b$, $Q_{\text{encl}} = \rho_l L$:

   $$\mathbf{E} = \hat{\boldsymbol{\rho}} \frac{\rho_l}{2\pi\epsilon_r\epsilon_0\rho}$$

   For $\rho > b$, $Q_{\text{encl}} = \rho_l L - \rho_l L = 0$, so $\mathbf{E} = 0$.

   (b) Find E at $\rho = a$ and at $\rho = b$.

   $$E(a) = \frac{2\times10^{-9}}{2\pi \times 2.1 \times 8.854\times10^{-12}\times10^{-3}} = 17.1 \text{ kV/m}$$
   $$E(b) = E(a)\frac{a}{b} = \frac{17.1}{3.5} = 4.89 \text{ kV/m}$$

   $E(a)$ = 17.1 kV/m

   $E(b)$ = 4.89 kV/m

   (c) For safety the field anywhere in the dielectric must stay below 20 MV/m. Find the largest ρ_l the line can carry, and say where the limit is reached first.

   The field is largest at $\rho = a$:

   $$\begin{aligned}\rho_{l,\max} &= 2\pi\epsilon_r\epsilon_0 a E_{\max} \\ &= 2\pi \times 2.1 \times 8.854\times10^{-12}\times10^{-3}\times 2\times10^{7} \\ &= 2.34\ \mu\text{C/m}\end{aligned}$$

   $\rho_{l,\max}$ = 2.34 μC/m

4. **Two parallel sheets.** In free space, an infinite sheet with $\rho_s = +1\ \mu\text{C/m}^2$ lies in the plane $z = 0$, and a second with $-\rho_s$ lies in the plane $z = d$.

   (a) Derive the field of a single infinite sheet from Gauss's law, using a pillbox.

   By symmetry **D** points away from a positive sheet on both sides with the same magnitude, $\mathbf{D} = \pm\hat{\mathbf{z}} D_z$. A pillbox with faces of area A on each side gets $D_z A$ from each face and nothing from its sides:

   $$2D_z A = \rho_s A \quad \Longrightarrow \quad \mathbf{E} = \pm\hat{\mathbf{z}}\,\frac{\rho_s}{2\epsilon_0}$$

   (b) Use superposition to find **E** between the sheets and outside them.

   Between the sheets, the positive sheet's field points in $+\hat{\mathbf{z}}$ (away from it) and the negative sheet's also points in $+\hat{\mathbf{z}}$ (toward it), so they add. Outside, they point in opposite directions and cancel:

   $$\mathbf{E} = \hat{\mathbf{z}}\,\frac{\rho_s}{\epsilon_0} = \hat{\mathbf{z}}\,\frac{10^{-6}}{8.854\times10^{-12}}$$
   $$= \hat{\mathbf{z}}\,113\ \text{kV/m}, \qquad 0 < z < d$$

   between = $\hat{\mathbf{z}}\,113$ kV/m

   outside = 0

   (c) This pair of sheets is the model for a parallel-plate capacitor. Explain why the field between them does not depend on d.

   The field of an infinite sheet is the same at every distance from it, so moving the sheets apart changes neither sheet's field, and their sum in the gap stays ρ_s/ϵ_0.

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