Learning Objective 5.1: I can apply Coulomb's Law to use direct integration to calculate the electric field due to charge density in various geometries, including rectangular, cylindrical, and spherical coordinate systems.

Show your work. Take $\epsilon = \epsilon_0 = 8.854 \times 10^{-12}$ F/m. Write a **Documentation** line at the end (**None** if you did not collaborate).

1. **A ring of charge.** A ring of radius a in the xy plane, centered on the origin, carries a uniform line charge ρ_ℓ. From the lesson, its field at height h on the axis is $\vec{E} = \hat{\mathbf{z}}\, \rho_\ell a h / 2\epsilon \left(a^2 + h^2\right)^{3/2}$.

   (a) With $a = 10$ cm and $\rho_\ell = 5$ nC/m, find the field at $h = 10$ cm.

   $E_z =$ 998 V/m

   $$E_z = \frac{(5 \times 10^{-9})(0.1)(0.1)}{2(8.854 \times 10^{-12})(0.02)^{3/2}} = \frac{5 \times 10^{-11}}{5.009 \times 10^{-14}} = 998 \text{ V/m}$$

   (b) The field is zero at the center of the ring and falls to zero far away. Find the height at which it is largest.

   $h =$ $a/\sqrt{2}$

   $$\frac{d}{dh}\left[\frac{h}{(a^2+h^2)^{3/2}}\right] = \frac{(a^2+h^2) - 3h^2}{(a^2+h^2)^{5/2}} = 0$$
   $$h = \frac{a}{\sqrt{2}} = 7.07 \text{ cm}$$

   (c) Now only the half of the ring with $0 \leqslant \phi' \leqslant \pi$ is charged. Derive the field at height h on the axis.

   $dQ = \rho_\ell\, a\, d\phi'$ and $\vec{R} = -a\hat{\boldsymbol{\rho}}' + h\hat{\mathbf{z}}$, as for the full ring. With $\hat{\boldsymbol{\rho}}' = \hat{\mathbf{x}}\cos\phi' + \hat{\mathbf{y}}\sin\phi'$,

   $$\int_0^\pi \hat{\boldsymbol{\rho}}'\, d\phi' = \hat{\mathbf{x}}\Big[\sin\phi'\Big]_0^\pi + \hat{\mathbf{y}}\Big[-\cos\phi'\Big]_0^\pi = 2\hat{\mathbf{y}}$$
   $$\vec{E} = \frac{\rho_\ell\, a}{4\pi\epsilon\,(a^2+h^2)^{3/2}}\left(-2a\hat{\mathbf{y}} + \pi h\hat{\mathbf{z}}\right)$$

   The x components still cancel in pairs across the y axis; the field leans toward $-\hat{\mathbf{y}}$, away from the charged half.

2. **A sheet with a hole.** An annulus in the xy plane, with inner radius b and outer radius a, carries a uniform surface charge ρ_s.

   (a) Derive the field at height $h > 0$ on the axis.

   $dQ = \rho_s\, \rho'\, d\rho'\, d\phi'$ and $\vec{R} = -\rho'\hat{\boldsymbol{\rho}}' + h\hat{\mathbf{z}}$. The radial terms integrate to zero over ϕ', as for the disk, and with $u = \rho'^2 + h^2$:

   $$E_z = \frac{\rho_s h}{4\pi\epsilon}\int_0^{2\pi} d\phi' \int_b^a \frac{\rho'\, d\rho'}{(\rho'^2+h^2)^{3/2}} = \frac{\rho_s h}{2\epsilon}\left[-\frac{1}{\sqrt{\rho'^2+h^2}}\right]_b^a = \frac{\rho_s h}{2\epsilon}\left(\frac{1}{\sqrt{b^2+h^2}} - \frac{1}{\sqrt{a^2+h^2}}\right)$$

   (b) Show that $b \to 0$ recovers the disk, and find the field of an infinite sheet with a hole of radius b.

   With $b = 0$ the first term is $1/h$, and $E_z = (\rho_s/2\epsilon)\left(1 - h/\sqrt{a^2+h^2}\right)$, the disk. With $a \to \infty$ the second term vanishes:

   $$E_z = \frac{\rho_s}{2\epsilon}\,\frac{h}{\sqrt{b^2+h^2}}$$

   (c) With $\rho_s = 1\ \mu$C/m^2 and $b = 20$ cm, find the field of the infinite sheet with a hole at $h = 20$ cm, and compare it with a sheet without one.

   E_z = 39.9 kV/m

   $$\frac{\rho_s}{2\epsilon} = \frac{1\times10^{-6}}{2(8.854\times10^{-12})} = 56.5\ \text{kV/m}$$
   $$E_z = 56.5 \times \frac{0.2}{\sqrt{0.08}} = 39.9\ \text{kV/m}$$

   The hole removes 29% of the field at this height.

3. **A thick spherical shell.** Charge fills the region $b < r < a$ with uniform density ρ_v; the region $r < b$ is empty.

(a) Using the lesson's result for a thin shell, derive the field in all three regions.

A thin shell of radius r' holds $dQ = \rho_v \, 4\pi r'^2 \, dr'$, acts as a point charge at the center for $r > r'$, and contributes nothing for $r < r'$. So only the charge between b and $\min(r, a)$ counts:

$$r < b: \quad E_r = 0$$

$$b < r < a: \quad E_r = \frac{1}{4\pi\epsilon r^2} \int_b^r \rho_v \, 4\pi r'^2 \, dr' = \frac{\rho_v \left(r^3 - b^3\right)}{3\epsilon r^2}$$

$$r > a: \quad E_r = \frac{1}{4\pi\epsilon r^2} \int_b^a \rho_v \, 4\pi r'^2 \, dr' = \frac{\rho_v \left(a^3 - b^3\right)}{3\epsilon r^2}$$

(b) With $\rho_v = 1\,\mu$C/m^3, $b = 1$ cm, and $a = 2$ cm, find the field at $r = 1.5$ cm and at $r = 4$ cm.

$$E_r(1.5\text{ cm}) = \frac{(1 \times 10^{-6})(2.375 \times 10^{-6})}{3(8.854 \times 10^{-12})(0.015)^2} = 397\text{ V/m}$$

$$E_r(4\text{ cm}) = \frac{(1 \times 10^{-6})(7 \times 10^{-6})}{3(8.854 \times 10^{-12})(0.04)^2} = 165\text{ V/m}$$

at 1.5 cm = 397 V/m

at 4 cm = 165 V/m

(c) Check part (a): what does it give when $b \to 0$, and when $b \to a$ with the total charge held fixed?

With $b = 0$ it is the solid sphere: $\rho_v r/3\epsilon$ inside and $\rho_v a^3/3\epsilon r^2$ outside. As $b \to a$ the charge becomes a thin shell: the middle region vanishes, the field is zero inside, and outside it is $Q/4\pi\epsilon r^2$ with $Q = \frac{4}{3}\pi\rho_v \left(a^3 - b^3\right)$.

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