Learning Objective 5.1: I can apply Coulomb's Law to use direct integration to calculate the electric field due to charge density in various geometries, including rectangular, cylindrical, and spherical coordinate systems.

Show your work. Take $\epsilon = \epsilon_0$ and $1/4\pi\epsilon_0 = 8.988 \times 10^9$ N·m^2/C^2. Write a **Documentation** line at the end (**None** if you did not collaborate).

1. **Superposition.** In free space, $q_1 = 4$ nC sits at the origin, $q_2 = -2$ nC at $(3, 0, 0)$ m, and $q_3 = 1$ nC at $(0, 4, 0)$ m.

   (a) Find the force on q_3.

$$\vec{R}_{31} = 4\hat{\mathbf{y}} \text{ m}, \quad |\vec{R}_{31}| = 4 \text{ m}$$
$$\vec{F}_{31} = \frac{(8.988 \times 10^9)(4 \times 10^{-9})(1 \times 10^{-9})}{4^3}(4\hat{\mathbf{y}})$$
$$= 2.247\hat{\mathbf{y}} \text{ nN}$$
$$\vec{R}_{32} = -3\hat{\mathbf{x}} + 4\hat{\mathbf{y}} \text{ m}, \quad |\vec{R}_{32}| = 5 \text{ m}$$
$$\vec{F}_{32} = \frac{(8.988 \times 10^9)(-2 \times 10^{-9})(1 \times 10^{-9})}{5^3} \times (-3\hat{\mathbf{x}} + 4\hat{\mathbf{y}})$$
$$= (0.431\hat{\mathbf{x}} - 0.575\hat{\mathbf{y}}) \text{ nN}$$
$$\vec{F}_3 = (0.431\hat{\mathbf{x}} + 1.672\hat{\mathbf{y}}) \text{ nN}$$

$\vec{F}_3 =$ $0.431\hat{\mathbf{x}} + 1.672\hat{\mathbf{y}}$ nN

   (b) Remove q_3. Find the electric field at $(0, 4, 0)$ m.

$$\vec{E} = \frac{\vec{F}_3}{q_3} = (0.431\hat{\mathbf{x}} + 1.672\hat{\mathbf{y}}) \text{ V/m}$$

$\vec{E} =$ $0.431\hat{\mathbf{x}} + 1.672\hat{\mathbf{y}}$ V/m

   (c) Find the point on the x axis, other than at infinity, where the field of q_1 and q_2 is zero. Explain why it cannot lie between the charges.

   Between the charges both fields point in $+\hat{\mathbf{x}}$, toward q_2, so they cannot cancel; to the left of q_1 the larger charge is always closer. So the point lies at $x > 3$ m, where the two fields point in opposite directions:

$$\frac{4}{x^2} = \frac{2}{(x-3)^2} \implies \sqrt{2}\,(x-3) = x$$
$$x = \frac{3\sqrt{2}}{\sqrt{2}-1} = 10.24 \text{ m}$$

$x =$ 10.24 m

2. **A dipole on its axis.** Charge $+q$ sits at $(0, 0, d/2)$ and $-q$ at $(0, 0, -d/2)$.

(a) Starting from the field of a point charge, derive the field at a point $(0, 0, z)$ on the axis with $z > d/2$.

For $+q$, $\vec{r} - \vec{r}' = \left(z - \frac{d}{2}\right)\hat{\mathbf{z}}$; for $-q$, $\vec{r} - \vec{r}' = \left(z + \frac{d}{2}\right)\hat{\mathbf{z}}$. Both point along $+\hat{\mathbf{z}}$, so

$$\vec{E} = \hat{\mathbf{z}}\frac{q}{4\pi\epsilon}\left[\frac{1}{\left(z - \frac{d}{2}\right)^2} - \frac{1}{\left(z + \frac{d}{2}\right)^2}\right] = \hat{\mathbf{z}}\frac{q}{4\pi\epsilon}\,\frac{2zd}{\left(z^2 - \frac{d^2}{4}\right)^2}$$

since $\left(z + \frac{d}{2}\right)^2 - \left(z - \frac{d}{2}\right)^2 = 2zd$.

(b) Show that for $z \gg d$ the field becomes $\hat{\mathbf{z}}\,2p/4\pi\epsilon z^3$, with $p = qd$. Compare it with the field at the same distance on the bisector, from the lesson.

For $z \gg d$, $z^2 - d^2/4 \approx z^2$:

$$\vec{E} \approx \hat{\mathbf{z}}\frac{q}{4\pi\epsilon}\,\frac{2zd}{z^4} = \hat{\mathbf{z}}\frac{2p}{4\pi\epsilon z^3}$$

On the bisector the field is $-\hat{\mathbf{y}}\,p/4\pi\epsilon x^3$: half as large, and pointing the opposite way, from $+q$ toward $-q$.

(c) With $q = 1$ nC and $d = 1$ cm, find the field at $z = 10$ cm from the exact result and from the approximation.

$$E_{\text{exact}} = (8.988 \times 10^9)(1 \times 10^{-9})\frac{2(0.1)(0.01)}{(0.01 - 0.000025)^2}$$

$$= 180.7 \text{ V/m}$$

$$E_{\text{approx}} = (8.988 \times 10^9)\frac{2(1 \times 10^{-11})}{(0.1)^3} = 179.8 \text{ V/m}$$

exact = 180.7 V/m

approx. = 179.8 V/m

3. **A line charge seen from one end.** A uniform line charge ρ_ℓ lies on the z axis from $z = 0$ to $z = L$. The field point is $(\rho, \phi, 0)$, a distance ρ from the axis in the plane of the line's lower end.

   (a) Derive both components of $\vec{E}$. Explain why the z component no longer vanishes.

   $dQ = \rho_\ell \, dz'$ and $\vec{R} = \rho\hat{\boldsymbol{\rho}} - z'\hat{\mathbf{z}}$, with $R = \sqrt{\rho^2 + z'^2}$. With $z' = \rho \tan\alpha$, as in the lesson,

   $$E_\rho = \frac{\rho_\ell \, \rho}{4\pi\epsilon} \int_0^L \frac{dz'}{(\rho^2 + z'^2)^{3/2}} = \frac{\rho_\ell \, \rho}{4\pi\epsilon} \left[\frac{z'}{\rho^2 \sqrt{\rho^2 + z'^2}} \right]_0^L = \frac{\rho_\ell}{4\pi\epsilon\rho} \frac{L}{\sqrt{\rho^2 + L^2}}$$

   $$E_z = -\frac{\rho_\ell}{4\pi\epsilon} \int_0^L \frac{z' \, dz'}{(\rho^2 + z'^2)^{3/2}} = -\frac{\rho_\ell}{4\pi\epsilon} \left[-\frac{1}{\sqrt{\rho^2 + z'^2}} \right]_0^L = -\frac{\rho_\ell}{4\pi\epsilon} \left(\frac{1}{\rho} - \frac{1}{\sqrt{\rho^2 + L^2}} \right)$$

   All the charge lies above the field point, so every element pushes it downward, along $-\hat{\mathbf{z}}$; there is no mirror element below to cancel it.

   (b) With $\rho_\ell = 2$ nC/m, $L = 1$ m, and $\rho = 0.5$ m, evaluate E_ρ and E_z.

   $\rho_\ell / 4\pi\epsilon = (2 \times 10^{-9})(8.988 \times 10^9) = 17.98$ V

   $$E_\rho = \frac{17.98}{0.5} \frac{1}{\sqrt{1.25}} = 32.2 \text{ V/m}$$

   $$E_z = -17.98\,(2 - 0.894) = -19.9 \text{ V/m}$$

   E_ρ = 32.2 V/m

   E_z = −19.9 V/m

   (c) The lesson's centered line of length L has field $\rho_\ell L / 4\pi\epsilon\rho\sqrt{\rho^2 + L^2/4}$. How long must it be, as a multiple of ρ, for its field to be within 1% of the infinite line's, $\rho_\ell / 2\pi\epsilon\rho$?

   The ratio of the two is $(L/2)/\sqrt{\rho^2 + L^2/4} = 0.99$:

   $$\left(\frac{L}{2}\right)^2 = 0.9801 \left(\rho^2 + \frac{L^2}{4}\right)$$

   $$\frac{L}{2} = \rho\sqrt{\frac{0.9801}{0.0199}} = 7.02\rho$$

   L = $14.0\,\rho$

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