Learning Objectives 4.1–4.3: I can apply the gradient, divergence, curl, and Laplacian in the three major coordinate systems; calculate surface and volume integrals in each; and apply the divergence theorem and Stokes' theorem to simplify integrals.

Show your work. Write a **Documentation** line at the end (**None** if you did not collaborate).

1. **Differential elements and integrals.**

   (a) Starting from the distance a point moves when each spherical coordinate changes by a small amount, derive the spherical volume element dv.

   Changing r by dr moves the point dr. Changing θ by $d\theta$ moves it along a circle of radius r, a distance $r\,d\theta$. Changing ϕ by $d\phi$ moves it along a horizontal circle of radius $r\sin\theta$, a distance $r\sin\theta\,d\phi$. The volume element is the product of the three edges:

   $$dv = (dr)(r\,d\theta)(r\sin\theta\,d\phi) = r^2\sin\theta\,dr\,d\theta\,d\phi$$

   (b) Find the area of the cap of a sphere of radius a that lies within $0 \leqslant \theta \leqslant \theta_0$. Check your answer at $\theta_0 = \pi$.

   $$S = \int_0^{2\pi}\int_0^{\theta_0} a^2\sin\theta\,d\theta\,d\phi = 2\pi a^2\,[-\cos\theta]_0^{\theta_0}$$
   $$= 2\pi a^2\,(1-\cos\theta_0)$$

   $S =$ $\boxed{2\pi a^2(1-\cos\theta_0)}$

   At $\theta_0 = \pi$, $S = 4\pi a^2$, the whole sphere.

   (c) Find the volume of a spherical shell with inner radius a and outer radius b.

   $$V = \int_0^{2\pi}\int_0^{\pi}\int_a^b r^2\sin\theta\,dr\,d\theta\,d\phi$$
   $$= 2\pi\cdot 2\cdot\frac{b^3-a^3}{3} = \frac{4\pi}{3}\left(b^3-a^3\right)$$

   $V =$ $\boxed{\frac{4\pi}{3}(b^3-a^3)}$

2. **Gradient, divergence, and the Laplacian.**

(a) Find the gradient of $f = xy$, and its magnitude at the point $(-4, 5, 0)$.

$$\nabla f = \hat{\mathbf{x}}\, y + \hat{\mathbf{y}}\, x = 5\hat{\mathbf{x}} - 4\hat{\mathbf{y}}$$
$$|\nabla f| = \sqrt{5^2 + 4^2} = \sqrt{41} = 6.40$$

$|\nabla f|$ = $\boxed{6.40}$

(b) Find the divergence of $\mathbf{F} = \hat{\boldsymbol{\rho}}\, \rho^2 + \hat{\mathbf{z}}\, z$.

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial\left(\rho \cdot \rho^2\right)}{\partial \rho} + \frac{\partial z}{\partial z} = \frac{3\rho^2}{\rho} + 1 = 3\rho + 1$$

$\nabla \cdot \mathbf{F}$ = $\boxed{3\rho + 1}$

(c) Find the Laplacian of $f = e^{xyz}$.

$$\frac{\partial^2 f}{\partial x^2} = y^2 z^2 e^{xyz}, \quad \text{and likewise in } y \text{ and } z$$
$$\nabla^2 f = \left(y^2 z^2 + x^2 z^2 + x^2 y^2\right) e^{xyz}$$

$\nabla^2 f$ = $\boxed{(y^2 z^2 + x^2 z^2 + x^2 y^2) e^{xyz}}$

(d) Find the Laplacian of $f = 1/(x^2 + y^2 + z^2)$. Use the coordinate system that makes it easiest.

In spherical coordinates $f = 1/r^2$, which depends only on r:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{-2}{r^3}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-2}{r}\right) = \frac{2}{r^4}$$

$\nabla^2 f$ = $\boxed{2/r^4}$

3. **The divergence theorem.** Verify the divergence theorem for $\mathbf{F} = \hat{\boldsymbol{\rho}}\,\rho^2 + \hat{\mathbf{z}}\,z$ over the cylinder $0 \leqslant \rho \leqslant a$, $0 \leqslant z \leqslant h$.

   (a) Evaluate the volume integral of $\nabla \cdot \mathbf{F}$.

   From Question 2(b), $\nabla \cdot \mathbf{F} = 3\rho + 1$:

$$\int_V \nabla \cdot \mathbf{F}\, dv = \int_0^h \int_0^{2\pi} \int_0^a (3\rho + 1)\, \rho\, d\rho\, d\phi\, dz$$
$$= 2\pi h \left(a^3 + \frac{a^2}{2} \right) = 2\pi a^3 h + \pi a^2 h$$

$$\boxed{2\pi a^3 h + \pi a^2 h}$$

   (b) Evaluate the flux of $\mathbf{F}$ out of each of the three surfaces of the cylinder, and show that the total equals your answer to part (a).

   Side, $\rho = a$: $d\mathbf{s} = \hat{\boldsymbol{\rho}}\, a\, d\phi\, dz$ and $F_\rho = a^2$:

$$\int_0^h \int_0^{2\pi} a^2 \cdot a\, d\phi\, dz = 2\pi a^3 h$$

   Top, $z = h$: $d\mathbf{s} = \hat{\mathbf{z}}\, \rho\, d\rho\, d\phi$ and $F_z = h$:

$$\int_0^{2\pi} \int_0^a h\, \rho\, d\rho\, d\phi = \pi a^2 h$$

   Bottom, $z = 0$: $d\mathbf{s} = -\hat{\mathbf{z}}\, \rho\, d\rho\, d\phi$ and $F_z = 0$, so the flux is 0. The total is $2\pi a^3 h + \pi a^2 h$, which matches part (a).

4. **Curl and Stokes' theorem.** The field $\mathbf{F} = \hat{\boldsymbol{\phi}}\,\rho$ circulates around the z axis.

   (a) Find $\nabla \times \mathbf{F}$.

   Only F_ϕ is nonzero, and it depends only on ρ, so only the z component survives:

   $$(\nabla \times \mathbf{F})_z = \frac{1}{\rho}\frac{\partial\,(\rho \cdot \rho)}{\partial \rho} = 2$$

   $\nabla \times \mathbf{F} =$ $\boxed{2\hat{\mathbf{z}}}$

   (b) Find the circulation of $\mathbf{F}$ counterclockwise around the circle $\rho = a$ in the xy plane, directly and with Stokes' theorem.

   Directly, $d\mathbf{l} = \hat{\boldsymbol{\phi}}\,a\,d\phi$ and $F_\phi = a$:

   $$\oint_C \mathbf{F} \cdot d\mathbf{l} = \int_0^{2\pi} a \cdot a\,d\phi = 2\pi a^2$$

   With Stokes' theorem, over the disk with $d\mathbf{s} = \hat{\mathbf{z}}\,\rho\,d\rho\,d\phi$:

   $$\int_S 2\,ds = 2 \cdot \pi a^2 = 2\pi a^2$$

   $\boxed{2\pi a^2}$

   (c) Find $\nabla \cdot \mathbf{F}$. In one or two sentences, explain what the divergence and the curl of this field tell you about it.

   $\nabla \cdot \mathbf{F} = (1/\rho)\,\partial F_\phi/\partial\phi = 0$. The field has no sources: every field line that enters a small volume also leaves it. Its curl is nonzero everywhere, so it circulates around every point, not only around the z axis.

   (d) State which theorem relates a volume integral to a closed-surface integral, and which relates an open-surface integral to a closed-path integral.

   The divergence theorem relates a volume to its closed surface: $\int_V \nabla \cdot \mathbf{F}\,dv = \oint_S \mathbf{F} \cdot d\mathbf{s}$. Stokes' theorem relates an open surface to its boundary path: $\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{s} = \oint_C \mathbf{F} \cdot d\mathbf{l}$.

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