Learning Objective 3.4: I can use the transmission line equation to match a load to a source using a length of transmission line.

Show your work. Write a **Documentation** line at the end (**None** if you did not collaborate). All lines are lossless, and d is the distance from the load toward the source.

1. **The matching point.** At a point on the line, the normalized impedance is $z = 1 + jx$.

   (a) Starting from $\Gamma = \frac{z-1}{z+1}$, show that $z = \frac{1+\Gamma}{1-\Gamma}$.

$$\begin{aligned} \Gamma(z+1) &= z-1 \\ z(1-\Gamma) &= 1+\Gamma \\ z &= \frac{1+\Gamma}{1-\Gamma} \end{aligned}$$

   (b) Show that at the point where $z = 1 + jx$,

$$x = \pm \frac{2|\Gamma|}{\sqrt{1-|\Gamma|^2}}.$$

$$\begin{aligned} \Gamma &= \frac{jx}{2+jx}, \qquad |\Gamma|^2 = \frac{x^2}{4+x^2} \\ |\Gamma|^2\left(4+x^2\right) &= x^2 \\ x^2\left(1-|\Gamma|^2\right) &= 4|\Gamma|^2 \\ x &= \pm \frac{2|\Gamma|}{\sqrt{1-|\Gamma|^2}} \end{aligned}$$

   (c) Explain in one sentence why $|\Gamma|$ in part (b) is the same as the magnitude of the load's reflection coefficient.

   Along a lossless line $\Gamma(d) = \Gamma e^{-j2\beta d}$, which changes only the phase of Γ, never its magnitude.

2. **A series match.** A $100\ \Omega$ load terminates a $50\ \Omega$ line at 2 GHz. Waves travel at 2×10^8 m/s in the line, so $\lambda = 10$ cm.

   (a) Find z_L, Γ, and the VSWR.

$$z_L = 2, \qquad \Gamma = \frac{2-1}{2+1} = \frac{1}{3}\angle 0^\circ$$

$$\text{VSWR} = \frac{4/3}{2/3} = 2$$

Γ = $0.333\angle 0^\circ$

VSWR = 2

   (b) Find the two values of x at the matching points, and the phase θ_d of Γ at each.

$$x = \pm\frac{2/3}{\sqrt{8/9}} = \pm 0.707$$

$$\theta_d = \pm\left(90^\circ - \tan^{-1} 0.354\right) = \pm\left(90^\circ - 19.5^\circ\right) = \pm 70.5^\circ$$

x = ± 0.707

θ_d = $\pm 70.5^\circ$

   (c) Find the distance d to each matching point, in wavelengths and in centimeters.

$$x = -0.707: \quad d = \frac{0^\circ - (-70.5^\circ)}{720^\circ}\lambda = 0.098\lambda = 0.98 \text{ cm}$$

$$x = +0.707: \quad d = \frac{0^\circ - 70.5^\circ + 360^\circ}{720^\circ}\lambda = 0.402\lambda = 4.02 \text{ cm}$$

d_1 = 0.098λ

d_2 = 0.402λ

   (d) Find the series component that completes each match, and its value.

   The reactance to cancel is $0.707 \times 50 = 35.4\ \Omega$. At d_1, $z = 1 - j0.707$, so we add a series inductor:

$$L = \frac{35.4}{2\pi \times 2 \times 10^9} = 2.81 \text{ nH}$$

   At d_2, $z = 1 + j0.707$, so we add a series capacitor:

$$C = \frac{1}{2\pi \times 2 \times 10^9 \times 35.4} = 2.25 \text{ pF}$$

at d_1: 2.81 nH

at d_2: 2.25 pF

   (e) Which design would you build? Justify your choice in one or two sentences.

   The first: its line is 0.098λ instead of 0.402λ, so its input impedance changes less with frequency and the match holds over a wider band.

3. **Checking a match.** Use the transmission line equation to check the first design from Question 2.

   (a) Find Z_{in} of the $100\,\Omega$ load seen through 0.098λ of $50\,\Omega$ line. ($\beta d = 35.3^\circ$ and $\tan \beta d = 0.707$.)

   Z_{in} = $50 - j35.4\,\Omega$

$$
\begin{aligned}
Z_{in} &= 50\,\frac{100 + j35.4}{50 + j70.7} \\
&= 50\,\frac{(100 + j35.4)(50 - j70.7)}{50^2 + 70.7^2} \\
&= 50\,\frac{7500 - j5300}{7500} = 50 - j35.4\,\Omega
\end{aligned}
$$

   (b) Add the 2.81 nH series inductor. What impedance does the source side see, and what is Γ there?

   Z = $50\,\Omega$

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\begin{aligned}
Z_{in} + j\omega L &= 50 - j35.4 + j35.4 = 50\,\Omega \\
\Gamma &= 0
\end{aligned}
$$

Documentation: _____