Learning Objectives 3.3 and 3.4: I can use the transmission line equation to calculate the voltage/current at an arbitrary point on a specified transmission line. I can use the transmission line equation to match a load to a source using a length of transmission line.

Show your work. Write a **Documentation** line at the end (**None** if you did not collaborate). All lines are lossless. The load is at $z = 0$ and d is the distance from the load toward the source.

1. **Deriving the input impedance.** The phasor voltage and current on a line terminated in Z_L are

$$\tilde{V}(z) = V_0^+ \left(e^{-j\beta z} + \Gamma e^{j\beta z}\right), \qquad \tilde{I}(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - \Gamma e^{j\beta z}\right)$$

(a) Write $Z_{in}(d) = \tilde{V}(-d)/\tilde{I}(-d)$, substitute $\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$, and show that

$$Z_{in}(d) = Z_0 \frac{Z_L + jZ_0 \tan\beta d}{Z_0 + jZ_L \tan\beta d}.$$

$$Z_{in} = Z_0 \frac{e^{j\beta d} + \Gamma e^{-j\beta d}}{e^{j\beta d} - \Gamma e^{-j\beta d}}$$

Multiply the numerator and denominator by $Z_L + Z_0$:

$$\begin{aligned} Z_{in} &= Z_0 \frac{(Z_L + Z_0)e^{j\beta d} + (Z_L - Z_0)e^{-j\beta d}}{(Z_L + Z_0)e^{j\beta d} - (Z_L - Z_0)e^{-j\beta d}} \\ &= Z_0 \frac{Z_L\left(e^{j\beta d} + e^{-j\beta d}\right) + Z_0\left(e^{j\beta d} - e^{-j\beta d}\right)}{Z_0\left(e^{j\beta d} + e^{-j\beta d}\right) + Z_L\left(e^{j\beta d} - e^{-j\beta d}\right)} \\ &= Z_0 \frac{2Z_L \cos\beta d + j2Z_0 \sin\beta d}{2Z_0 \cos\beta d + j2Z_L \sin\beta d} \end{aligned}$$

using Euler's identity. Dividing the numerator and denominator by $2\cos\beta d$ gives the result.

(b) Set $Z_L = 0$ and then $Z_L \to \infty$ to find the input impedance of a shorted line and of an open line.

$$\begin{aligned} Z_L = 0: \quad & Z_{in} = Z_0 \frac{jZ_0 \tan\beta d}{Z_0} = jZ_0 \tan\beta d \\ Z_L \to \infty: \quad & Z_{in} = Z_0 \frac{Z_L}{jZ_L \tan\beta d} = -jZ_0 \cot(\beta)\, d \end{aligned}$$

For the open line we divide the numerator and denominator by Z_L before taking the limit.

2. **One load, three lengths.** A $100\,\Omega$ resistor terminates a $50\,\Omega$ line.

   (a) Find Z_{in} for a line $\lambda/8$ long.

   $\beta d = 45°$, so $\tan \beta d = 1$:

   $$Z_{in} = 50\,\frac{100 + j50}{50 + j100} = 50\,\frac{(100 + j50)(50 - j100)}{50^2 + 100^2}$$
   $$= 50\,\frac{10{,}000 - j7{,}500}{12{,}500} = 40 - j30\,\Omega$$

   Z_{in} = $40 - j30\,\Omega$

   (b) Find Z_{in} for lines $\lambda/4$ and $\lambda/2$ long.

   $$\lambda/4: \quad Z_{in} = \frac{Z_0^2}{Z_L} = \frac{2500}{100} = 25\,\Omega$$
   $$\lambda/2: \quad Z_{in} = Z_L = 100\,\Omega$$

   $\lambda/4$: $25\,\Omega$

   $\lambda/2$: $100\,\Omega$

   (c) The resistor absorbs real power, but the input impedance in part (a) has a reactive part. Explain in one or two sentences how both can be true.

   The reactance in Z_{in} comes from the standing wave on the line, which stores energy that flows back and forth between the line's inductance and capacitance. The lossless line dissipates nothing, so the real power that enters at the input is the real power the resistor absorbs.

3. **Stubs.** A circuit at 2 GHz needs a 5 nH inductor and a 1 pF capacitor. Build each from $50\,\Omega$ line in which waves travel at 2×10^8 m/s.

   (a) Find the wavelength in the line.

$$\lambda = \frac{2 \times 10^8}{2 \times 10^9} = 0.1 \text{ m} = 10 \text{ cm}$$

$\lambda =$ 10 cm

   (b) Find the length of the shortest shorted stub that acts as the inductor.

$$\omega L = 2\pi \times 2 \times 10^9 \times 5 \times 10^{-9} = 62.8\,\Omega$$
$$\tan \beta\ell = \frac{62.8}{50} = 1.257 \quad \Rightarrow \quad \beta\ell = 51.5^\circ$$
$$\ell = \frac{51.5^\circ}{360^\circ}\lambda = 0.143\lambda = 1.43 \text{ cm}$$

$\ell =$ 1.43 cm

   (c) Find the length of the shortest open stub that acts as the capacitor.

$$\frac{1}{\omega C} = \frac{1}{2\pi \times 2 \times 10^9 \times 10^{-12}} = 79.6\,\Omega$$
$$\cot(\beta)\,\ell = \frac{79.6}{50} = 1.592 \quad \Rightarrow \quad \beta\ell = 32.1^\circ$$
$$\ell = \frac{32.1^\circ}{360^\circ}\lambda = 0.089\lambda = 0.89 \text{ cm}$$

$\ell =$ 0.89 cm

   (d) Your stub in part (b) is accidentally cut 2.5 cm longer than designed. What does it act like now? Explain without computing.

   2.5 cm is $\lambda/4$, and a shorted line between $\lambda/4$ and $\lambda/2$ long has $\tan \beta\ell < 0$: it acts as a capacitor.

4. **Quarter-wave transformer.** A $200\ \Omega$ antenna is fed at 500 MHz from a $50\ \Omega$ line. Waves travel at 2×10^8 m/s in every line.

   (a) Find the characteristic impedance and the length of the quarter-wave section that matches the antenna.

$$Z_1 = \sqrt{Z_0\, R_L} = \sqrt{50 \times 200} = 100\ \Omega$$
$$\lambda = \frac{2 \times 10^8}{5 \times 10^8} = 40\ \text{cm}, \quad \frac{\lambda}{4} = 10\ \text{cm}$$

Z_1 = $100\ \Omega$

length = 10 cm

   (b) The same network is used at 1 GHz. Find the impedance the $50\ \Omega$ line sees and the reflection coefficient there.

   At 1 GHz the 10 cm section is $\lambda/2$ long, so it repeats the load:

$$Z_{in} = 200\ \Omega$$
$$\Gamma = \frac{200 - 50}{200 + 50} = 0.6$$

Z_{in} = $200\ \Omega$

Γ = 0.6

5. **Power delivered.** An incident wave of amplitude $|V_0^+| = 10$ V travels on a $50\,\Omega$ line toward a $100\,\Omega$ load.

(a) Find the incident power.

$$P_{\text{inc}} = \frac{|V_0^+|^2}{2Z_0} = \frac{100}{100} = 1\text{ W}$$

P_{inc} = 1 W

(b) Find the power the load absorbs.

$$\Gamma = \frac{100-50}{100+50} = \frac{1}{3}$$
$$P_L = P_{\text{inc}}\left(1-|\Gamma|^2\right) = 1 \times \frac{8}{9} = 0.889\text{ W}$$

P_L = 0.889 W

(c) Check part (b) directly: find the load voltage $\tilde{V}(0) = V_0^+(1+\Gamma)$ and compute $\frac{1}{2}|\tilde{V}(0)|^2/R_L$.

$$|\tilde{V}(0)| = 10 \times \frac{4}{3} = 13.3\text{ V}$$
$$P_L = \frac{13.3^2}{2 \times 100} = 0.889\text{ W}$$

P_L = 0.889 W

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