Learning Objective 3.1: I can describe the origin of the transmission line equation, understand the method of developing a solution, and interpret the solution.

Show your work. Write a **Documentation** line at the end (**None** if you did not collaborate).

1. **From the telegrapher's equations to the wave equation.** Start from the telegrapher's equations,

$$-\frac{\partial v(z,t)}{\partial z} = R'\, i(z,t) + L'\frac{\partial i(z,t)}{\partial t}$$
$$-\frac{\partial i(z,t)}{\partial z} = G'\, v(z,t) + C'\frac{\partial v(z,t)}{\partial t}$$

(a) With $v(z,t) = \text{Re}\left[\tilde{V}(z)e^{j\omega t}\right]$, show that the phasor of $\partial v/\partial t$ is $j\omega\tilde{V}(z)$.

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\text{Re}\left[\tilde{V}(z)e^{j\omega t}\right] = \text{Re}\left[\tilde{V}(z)\frac{\partial}{\partial t}e^{j\omega t}\right] = \text{Re}\left[j\omega\tilde{V}(z)e^{j\omega t}\right]$$

The derivative passes inside the real part because t is real, and $\tilde{V}(z)$ does not depend on t. The result is the real part of $j\omega\tilde{V}(z)\,e^{j\omega t}$, so its phasor is $j\omega\tilde{V}(z)$.

(b) Write both telegrapher's equations in phasor form.

$$-\frac{d\tilde{V}(z)}{dz} = \left(R' + j\omega L'\right)\tilde{I}(z)$$
$$-\frac{d\tilde{I}(z)}{dz} = \left(G' + j\omega C'\right)\tilde{V}(z)$$

(c) Uncouple the equations to derive the wave equation for the current, $\tilde{I}(z)$. Show each step.

Differentiate the second equation with respect to z, then substitute the first for $d\tilde{V}/dz$:

$$-\frac{d^2\tilde{I}(z)}{dz^2} = \left(G' + j\omega C'\right)\frac{d\tilde{V}(z)}{dz}$$
$$= -\left(G' + j\omega C'\right)\left(R' + j\omega L'\right)\tilde{I}(z)$$
$$\frac{d^2\tilde{I}(z)}{dz^2} - \gamma^2\tilde{I}(z) = 0, \qquad \gamma^2 = \left(R' + j\omega L'\right)\left(G' + j\omega C'\right)$$

(d) Substitute the trial solution $\tilde{I}(z) = Ae^{sz}$ into your result, find the characteristic equation and its roots, and write the general solution.

$$s^2Ae^{sz} - \gamma^2Ae^{sz} = 0$$
$$s^2 - \gamma^2 = 0 \quad \Longrightarrow \quad s = \pm\gamma$$
$$\tilde{I}(z) = I_0^+e^{-\gamma z} + I_0^-e^{+\gamma z}$$

2. **The propagation constant of a lossy line.** A line has $R' = 0.5\ \Omega/\text{m}$, $L' = 250$ nH/m, $G' = 0$, and $C' = 100$ pF/m, and carries a 10 MHz signal.

   (a) Find the series impedance per meter, $R' + j\omega L'$, and the shunt admittance per meter, $G' + j\omega C'$.

$$\omega = 2\pi \times 10^7 = 6.283 \times 10^7 \text{ rad/s}$$
$$R' + j\omega L' = 0.5 + j15.71\ \Omega/\text{m}$$
$$G' + j\omega C' = j6.283 \times 10^{-3} \text{ S/m}$$

Z' = $0.5 + j15.71\ \Omega/\text{m}$

Y' = $j6.28 \times 10^{-3}$ S/m

   (b) Find $\gamma = \alpha + j\beta$.

$$\gamma^2 = (0.5 + j15.71)(j6.283 \times 10^{-3})$$
$$= -0.0987 + j0.00314$$
$$= 0.0987\angle 178.18^\circ$$
$$\gamma = \sqrt{0.0987}\angle 89.09^\circ = 0.3142\angle 89.09^\circ$$
$$= 0.0050 + j0.3142\ \text{m}^{-1}$$

α = 0.0050 Np/m

β = 0.314 rad/m

   We take the root with $\alpha \geqslant 0$ and $\beta > 0$.

   (c) Find the wavelength on the line.

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.3142} = 20.0 \text{ m}$$

λ = 20.0 m

   (d) Compare β with its value on the same line with $R' = 0$. What does the comparison say about the effect of a small loss on the phase constant?

   With $R' = 0$, $\beta = \omega\sqrt{L'C'} = 6.283 \times 10^7\sqrt{2.5 \times 10^{-17}} = 0.3142$ rad/m, the same to four figures. A small loss adds attenuation but leaves the phase constant, and so the wavelength, essentially unchanged.

3. **Reading a solution.** The voltage on a line is

$$v(z,t) = 5\,e^{-0.02z}\cos\left(2\pi \times 10^8\,t - 3z + 30^\circ\right) + 2\,e^{0.02z}\cos\left(2\pi \times 10^8\,t + 3z\right)\ \text{V}$$

with z in meters.

(a) Which term travels toward $+z$ and which toward $-z$? Explain how you can tell.

The first term travels toward $+z$: a point of constant phase, $\omega t - 3z =$ constant, moves to larger z as t grows. The second travels toward $-z$, because holding $\omega t + 3z$ constant requires z to decrease as t grows.

(b) Find α, β, and the wavelength.

$\alpha = 0.02$ Np/m, $\beta = 3$ rad/m, $\lambda = \dfrac{2\pi}{3} = 2.09$ m

α = 0.02 Np/m

λ = 2.09 m

(c) Write the phasor constants V_0^+ and V_0^-.

$V_0^+ = 5\angle 30^\circ$ V, $V_0^- = 2\angle 0^\circ$ V

V_0^+ = $5\angle 30^\circ$ V

V_0^- = $2\angle 0^\circ$ V

(d) The second term grows as $e^{0.02z}$. Explain why this does not mean the reflected wave gains energy.

The reflected wave travels toward $-z$. Following it in its own direction of travel, z decreases, so $e^{0.02z}$ decreases: the wave decays as it travels, just as the incident wave does.

4. **The lossless line.** RG-58 has $L' = 259$ nH/m and $C' = 96.8$ pF/m.

   (a) Set $R' = G' = 0$ and show that γ is purely imaginary.

$$\gamma = \sqrt{(j\omega L')(j\omega C')} = \sqrt{-\omega^2 L'C'} = j\omega\sqrt{L'C'}$$

   so $\alpha = 0$ and $\beta = \omega\sqrt{L'C'}$.

   (b) Find β and the wavelength at 50 MHz.

$$\beta = 2\pi \times 5 \times 10^7 \sqrt{259 \times 10^{-9} \times 96.8 \times 10^{-12}}$$
$$= 1.57 \text{ rad/m}$$
$$\lambda = \frac{2\pi}{1.57} = 4.0 \text{ m}$$

β = 1.57 rad/m

λ = 4.0 m

   (c) In the time domain, describe what the incident wave on a lossless line looks like to an observer at a fixed position z, and to an observer moving along with one of its crests.

   At a fixed z, the voltage is a sinusoid in time with constant amplitude $|V_0^+|$. An observer moving with a crest sees a constant voltage, because the phase $\omega t - \beta z$ does not change, and with $\alpha = 0$ the amplitude does not change either.

Documentation: _____