Learning Objective 3.1: I can describe the origin of the transmission line equation, understand the method of developing a solution, and interpret the solution.

Show your work. Write a **Documentation** line at the end (**None** if you did not collaborate). Unless a problem says otherwise, signals travel along RG-58 coaxial cable at $v_p = 2 \times 10^8$ m/s.

1. **Electrical length.** A 3 m length of RG-58 connects a signal generator to a load.

   (a) Find the wavelength in the cable at 50 kHz and the electrical length ℓ/λ. Does the cable behave as an ideal wire?

   $$\lambda = \frac{v_p}{f} = \frac{2 \times 10^8}{5 \times 10^4} = 4000 \text{ m}$$
   $$\frac{\ell}{\lambda} = \frac{3}{4000} = 7.5 \times 10^{-4}$$

   The cable is far shorter than $\lambda/10$, so it behaves as an ideal wire.

   λ = 4 km

   ℓ/λ = 7.5×10^{-4}

   (b) Repeat part (a) at 150 MHz.

   $$\lambda = \frac{2 \times 10^8}{1.5 \times 10^8} = 1.33 \text{ m}$$
   $$\frac{\ell}{\lambda} = \frac{3}{1.33} = 2.25$$

   The cable is more than two wavelengths long. We must treat it as a transmission line.

   λ = 1.33 m

   ℓ/λ = 2.25

   (c) At 150 MHz, find the phase delay from one end of the cable to the other, in degrees, and the time delay.

   $$\Delta\phi = 360^\circ \cdot \frac{\ell}{\lambda} = 360^\circ \times 2.25 = 810^\circ$$
   $$\tau = \frac{\ell}{v_p} = \frac{3}{2 \times 10^8} = 15 \text{ ns}$$

   The time delay does not depend on frequency; at 150 MHz it spans 2.25 periods.

   $\Delta\phi$ = 810°

   τ = 15 ns

   (d) Using the $\lambda/10$ rule of thumb, find the highest frequency at which this cable still behaves as an ideal wire.

   $$\lambda = 10\ell = 30 \text{ m}$$
   $$f = \frac{v_p}{\lambda} = \frac{2 \times 10^8}{30} = 6.67 \text{ MHz}$$

   f_{max} = 6.67 MHz

2. **Wire or transmission line?**

(a) A microstrip trace on a circuit board is 5 cm long and carries a 1 GHz clock. Signals on the board travel at $v_p = 1.65 \times 10^8$ m/s. Find ℓ/λ and decide which model applies.

$\ell/\lambda =$ 0.30

$$\lambda = \frac{1.65 \times 10^8}{10^9} = 16.5 \text{ cm}$$
$$\frac{\ell}{\lambda} = \frac{5}{16.5} = 0.30$$

At 0.30λ the trace is three times past the rule of thumb, so it is a transmission line.

(b) A 10 m speaker cable carries audio up to 20 kHz at $v_p = 2 \times 10^8$ m/s. Find ℓ/λ at the highest audio frequency and decide which model applies.

$\ell/\lambda =$ 10^{-3}

$$\lambda = \frac{2 \times 10^8}{2 \times 10^4} = 10 \text{ km}$$
$$\frac{\ell}{\lambda} = \frac{10}{10^4} = 10^{-3}$$

The cable is a thousandth of a wavelength, so it behaves as an ideal wire.

(c) The speaker cable in part (b) is 200 times longer than the circuit-board trace in part (a). Explain in one or two sentences why the short trace needs the transmission line model and the long cable does not.

What sets the model is the length measured in wavelengths, not in meters. The 1 GHz clock has a wavelength of 16.5 cm, so the trace holds a third of a cycle, while the audio signal has a wavelength of 10 km, so the cable holds a thousandth of one.

3. **The four elements.** The lumped-element model places $R'\Delta z$ and $L'\Delta z$ in series and $G'\Delta z$ and $C'\Delta z$ in shunt in each section of the line.

   (a) For each of R', L', G', and C', give its units and the physical effect in the line that it represents.

   R' (Ω/m) represents the resistance of the metal conductors. L' (H/m) represents the energy stored in the magnetic field around the current. G' (S/m) represents the leakage current through the imperfect dielectric. C' (F/m) represents the energy stored in the electric field between the conductors.

   (b) Which elements are zero on a lossless line, and why?

   $R' = 0$ and $G' = 0$. Both represent power dissipated in the line, in the conductors and in the dielectric. A lossless line stores energy in L' and C' but dissipates none.

   (c) RG-58 has $L' = 259$ nH/m and $C' = 96.8$ pF/m. Find the total inductance and capacitance of a 1.5 m length.

   $$L = L'\ell = 259 \times 1.5 = 389 \text{ nH}$$
   $$C = C'\ell = 96.8 \times 1.5 = 145 \text{ pF}$$

   L = 389 nH

   C = 145 pF

   (d) A classmate claims that $G' = 1/R'$, since conductance is the inverse of resistance. Explain why the claim is wrong.

   R' and G' describe two different materials in two different places. R' is the resistance of the metal along the conductors, in series with the current, and G' is the conductance of the dielectric between them, in shunt. A good line has small R' and small G' at the same time, so the two are independent.

4. **Kirchhoff's laws on one section.** Use the section in the lesson, with $v(z,t)$ and $i(z,t)$ at the input and $v(z+\Delta z,t)$ and $i(z+\Delta z,t)$ at the output, and the shunt elements connected at the output side of the series elements.

   (a) Write Kirchhoff's voltage law around the loop from the input terminals to the output terminals.

$$v(z,t) - R'\Delta z\, i(z,t) - L'\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z,t) = 0$$

   (b) Write Kirchhoff's current law at the node where the shunt elements connect.

$$i(z,t) - G'\Delta z\, v(z+\Delta z,t) - C'\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0$$

   (c) We choose $\Delta z = \lambda/20$ for a model of RG-58 at 500 MHz. Find Δz and the number of sections needed to model a 1 m cable.

$$\lambda = \frac{2\times 10^8}{5\times 10^8} = 0.4 \text{ m}$$
$$\Delta z = \frac{0.4}{20} = 2 \text{ cm}$$
$$N = \frac{1}{0.02} = 50$$

Δz =	2 cm
N =	50

   (d) Starting from your result in part (a), divide by Δz and take the limit as $\Delta z \to 0$ to derive the first telegrapher's equation. Name the step where the derivative in z appears.

   Move the two voltages to one side and divide by Δz:

$$\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = -R'\, i(z,t) - L' \frac{\partial i(z,t)}{\partial t}$$

   The right side no longer depends on Δz. As $\Delta z \to 0$, the left side is the definition of the partial derivative with respect to z:

$$-\frac{\partial v(z,t)}{\partial z} = R'\, i(z,t) + L' \frac{\partial i(z,t)}{\partial t}$$

5. **The telegrapher's equations.** The two telegrapher's equations for a line with all four elements are

$$-\frac{\partial v(z,t)}{\partial z} = R'\, i(z,t) + L'\frac{\partial i(z,t)}{\partial t}$$
$$-\frac{\partial i(z,t)}{\partial z} = G'\, v(z,t) + C'\frac{\partial v(z,t)}{\partial t}$$

(a) In the current law for one section, the shunt elements see $v(z+\Delta z, t)$, not $v(z,t)$. Explain why the second equation contains $v(z,t)$.

As $\Delta z \to 0$ the points z and $z+\Delta z$ merge, so $v(z+\Delta z, t) \to v(z,t)$. Placing the shunt elements before or after the series elements changes the equation for one finite section, but not the limit.

(b) Write both equations for a lossless line.

$$-\frac{\partial v(z,t)}{\partial z} = L'\frac{\partial i(z,t)}{\partial t}$$
$$-\frac{\partial i(z,t)}{\partial z} = C'\frac{\partial v(z,t)}{\partial t}$$

(c) The equations are called *coupled*. Explain what that means, and why we cannot solve the first equation for $v(z,t)$ by itself.

Each equation contains both v and i: the change in voltage along the line depends on the current, and the change in current depends on the voltage. The first equation has two unknown functions, so it cannot determine v without the second. Lesson 4 combines them into one equation for v alone.

(d) On a lossless line carrying a DC voltage and current that do not change in time, what do the equations say about v and i along the line? Explain the result physically.

Every time derivative is zero, so $\partial v/\partial z = 0$ and $\partial i/\partial z = 0$: the voltage and current are the same everywhere on the line. At DC the inductance has no voltage across it and the capacitance draws no current, so a lossless line behaves as an ideal pair of wires, as circuit theory assumes.

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