ECE343 Spring 2025

Objective 9.1: Helmholtz Wave Equations



UNITED STATES
AIR FORCE
ACADEMY

SCHEDULE

Lesson	Topics	Reading	Notes
24	BVP (Capacitor, Coax), Electric Potential	5.16	LO 5.3
25	Electric Boundary Conditions	5.18-5.22	LO 5.4
26	BC's/Capacitance	5.23-5.24	
27	Electrostatic Lab	Lab packet	
Chapter 6: Steady Current & Conductivity			
28	Current density, Conductivity, Resistance	Ch. 6	
Chapter 7: Magnetostatics			
29	Magnetic Fields, Gauss' Law for Magnetism	7.1-7.3	LO 7.1-7.2 **GR 2**
30	Ampere's Law	7.4-7.5, 7.9	LO 7.3
31	Ampere's Law Applied	7.6-7.8	LO 7.3
32	Magnetic Boundary Conditions	7.10-7.14	LO 7.4
33	Magnetostatics Lab	Lab packet	
Chapter 8: Time-Varying Fields			
34	Faraday's Law, Displacement Current, Maxwell Eqns	8.3, 8.8, 8.9	LO 8.1
35	Faraday's Law, Displacement Current, Maxwell Eqns	8.3, 8.8, 8.9	LO 8.2-8.3
Chapter 9: Plane Waves in Lossless Media			
36	Maxwell's Equations in lossless, source-free region	9.1-9.2	LO 9.1
37	Maxwell's Equations in lossless, source-free region	9.1-9.2	LO 9.1
38	Uniform Plane Waves & Polarization	9.4-9.6	LO 9.2-9.3
39	Normal Incidence	Vol 2, 5.6-5.7	LO 9.4-9.5
40	Oblique Incidence	Vol 2, 5.6-5.7	LO 9.5
41	Oblique Incidence	Vol 2, 5.6-5.7	LO 9.5
42	Incidence Lab	TBD	LO 9.5

Ch 9 Learning Objectives

9.1 I understand the process of decoupling and finding solutions for Maxwell's equations for dynamic fields.

9.2 I can articulate what a Uniform Plane Wave (UPW) is and describe key properties that relate the electric and magnetic field.

9.3 I can calculate the electric and magnetic fields for a UPW.

9.4 I can articulate the concept of polarization and determine the polarization for various UPWs.

9.5 I can determine the reflection and transmission of UPW at a boundary for Transverse Electric (TE) and Transverse Magnetic (TM) waves, using boundary conditions.

Maxwell's Equations

Integral Form	Point Form
$\iint \epsilon \vec{E} \cdot d\vec{S} = Q_{enc}$	$\nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon}$
$\oint \vec{H} \cdot d\vec{S} = 0$	$\nabla \cdot \vec{H} = 0$
$\oint_C \vec{E} \cdot d\ell = -\frac{\partial \Psi_m}{\partial t}$ $\oint_C \vec{E} \cdot d\ell = -\iint \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$	$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$
$\oint_C \vec{H} \cdot d\ell = I_{encl}$ $\oint_C \vec{H} \cdot d\ell = \int \vec{J} \cdot d\vec{S} + \epsilon \frac{\partial}{\partial t} \iint \vec{E} \cdot d\vec{S}$	$\nabla \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}$

coupled PDE's

Maxwell's Equations: Phasors

1. Starting with Gauss' Law:

$$\nabla \cdot \vec{E} = \rho_v$$

2. Assume that the electric and magnetic field are sinusoidally varying:

$$\vec{E}, \vec{H} \sim E_0, H_0 \cos(\omega t - \beta z + \phi)$$

3. Using a phasor representation of electric flux density and charge density:

$$\vec{E} = \mathrm{Re}\left[\tilde{E}_0 e^{j\omega t}\right]$$

$$\frac{d\vec{E}}{dt} \rightarrow j\omega \tilde{E}$$

$$\tilde{E} \sim e^{-j(\beta z - \phi)}$$

4. What is the phasor form of Gauss' Law?

$$\nabla \cdot \left[\mathrm{Re}\left[\tilde{E} e^{j\omega t}\right]\right] = \frac{1}{\epsilon} \mathrm{Re}\left[\tilde{\rho}_v e^{j\omega t}\right]$$

$$\mathrm{Re}\left[\nabla \cdot \left[\tilde{E} e^{j\omega t}\right]\right] = \frac{1}{\epsilon} \mathrm{Re}\left[\tilde{\rho}_v e^{j\omega t}\right]$$

$$\mathrm{Re}\left[(\nabla \cdot \tilde{E}) e^{j\omega t}\right] = \frac{1}{\epsilon}$$

$$\nabla \cdot \tilde{E} = \frac{\tilde{\rho}_v}{\epsilon}$$

Gauss' Law

$$\nabla \cdot \tilde{E} = \frac{\tilde{\rho}_v}{\epsilon}$$

Faraday's Law

1. Using the same approach, let's look at Faraday's Law:

2. Assume that the electric and magnetic field are sinusoidally varying:

3. Using the phasor representations:

4. What is the phasor form of Faraday's Law?

Faraday's Law

Maxwell's Equations: Phasor Forms

General

$$\nabla \cdot \tilde{E} = \frac{\tilde{\rho}_v}{\varepsilon}$$

$$\nabla \cdot \tilde{B} = 0$$

$$\nabla \times \tilde{H} = \tilde{J} + j\omega\varepsilon\tilde{E} = (\sigma + j\omega\varepsilon)\,\tilde{E}$$

$$\nabla \times \tilde{E} = -j\omega\mu\tilde{H}$$

Charge-free Region

$$\nabla \cdot \tilde{E} = 0$$

$$\nabla \cdot \tilde{B} = 0$$

$$\nabla \times \tilde{H} = j\omega\varepsilon\tilde{E}$$

$$\nabla \times \tilde{E} = -j\omega\mu\tilde{H}$$

$$\nabla \times \tilde{H} = \left(\underbrace{\sigma}_{\text{Real}} + \underbrace{j\omega\varepsilon}_{\text{imag}}\right)\tilde{E} \Longrightarrow \underbrace{\tan(\delta) = \frac{\sigma}{\omega\varepsilon}}_{\text{Loss tangent}}$$

Wave Equations

The phasor form of Maxwell-Faraday and Maxwell-Ampere's equations are **coupled**, **partial differential equations**. To find a solution:

- Use phasors to find time-independent expressions
- Decouple equations
- Solve resulting 2nd order ODEs using normal methods
- Interpret results

Phasor Forms

$$\nabla \times \tilde{E} = -j\omega\mu\tilde{H} \qquad \nabla \times \tilde{H} = (\sigma + j\omega\varepsilon)\,\tilde{E}$$

Decoupling Maxwell's Equations

$\nabla^2 \tilde{E} - \gamma^2 \tilde{E} = 0$

Starting with Faraday's law:

$\nabla \times \tilde{E} = -j\omega\mu \tilde{H}$

$\nabla \times \nabla \times \tilde{E} = \nabla \times (-j\omega\mu \tilde{H})$

$\nabla(\nabla \cdot \tilde{E}) - \nabla^2 \tilde{E} = -j\omega\mu(\nabla \times \tilde{H})$

↓ Gauss (source-free)

Amp.

0

$\nabla^2 \tilde{E} = j\omega\mu\{j\omega\epsilon \tilde{E}\}$

$\nabla^2 \tilde{E} + \omega^2\mu\epsilon \tilde{E} = 0$

General.. $\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon) = -\omega^2\mu\epsilon$

Your turn – decouple starting with Ampere's Law...

Helmholtz Wave Equations

$$\overleftrightarrow{\nabla}^2 = \hat{x}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) + \hat{y}\left(\quad\right) + \hat{z}\left(\frac{\partial^2}{\partial x}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$$

$$\frac{\partial^2}{\partial z^2}\tilde{E}_z - \gamma\tilde{E}_z$$

$$\overleftrightarrow{\nabla}^2\tilde{E} - \gamma^2\tilde{E} = 0$$

$$\overleftrightarrow{\nabla}^2\tilde{H} - \gamma^2\tilde{H} = 0$$

$$\gamma^2 = j\omega\mu(\sigma' + j\omega\epsilon)$$