ECE343 Spring 2025

Objective 5.2:
Gauss' Law



UNITED STATES
AIR FORCE
ACADEMY

Lesson	Topics	Reading	Notes
24	BVP (Capacitor, Coax), Electric Potential	5.16	LO 5.3
25	Electric Boundary Conditions	5.18-5.22	LO 5.4
26	BC's/Capacitance	5.23-5.24	
27	Electrostatic Lab	Lab packet	
Chapter 6: Steady Current & Conductivity			
28	Current density, Conductivity, Resistance	Ch. 6	
Chapter 7: Magnetostatics			
29	Magnetic Fields, Gauss' Law for Magnetism	7.1-7.3	LO 7.1-7.2 **GR 2**
30	Ampere's Law	7.4-7.5, 7.9	LO 7.3
31	Ampere's Law Applied	7.6-7.8	LO 7.3
32	Magnetic Boundary Conditions	7.10-7.14	LO 7.4
33	Magnetostatics Lab	Lab packet	
Chapter 8: Time-Varying Fields			
34	Faraday's Law, Displacement Current, Maxwell Eqns	8.3, 8.8, 8.9	LO 8.1
35	Faraday's Law, Displacement Current, Maxwell Eqns	8.3, 8.8, 8.9	LO 8.2-8.3
Chapter 9: Plane Waves in Lossless Media			
36	Maxwell's Equations in lossless, source-free region	9.1-9.2	LO 9.1
37	Maxwell's Equations in lossless, source-free region	9.1-9.2	LO 9.1
38	Uniform Plane Waves & Polarization	9.4-9.6	LO 9.2-9.3
39	Normal Incidence	Vol 2, 5.6-5.7	LO 9.4-9.5
40	Oblique Incidence	Vol 2, 5.6-5.7	LO 9.5
41	Oblique Incidence	Vol 2, 5.6-5.7	LO 9.5
42	Incidence Lab	TBD	LO 9.5

Ch 5 Learning Objectives

5.1 I can apply Coloumb's Law to use direct integration to calculate the electric field due to charge density in varoius geometries, including rectangular, cylindrical, and spherical coordinate systems.

5.2 I can apply the integral and differential forms of Gauss' Law to calculate the electric field due to charge density in various geometries, including rectangular, cylindrical, and spherical coordinate systems.

5.3 I can calculate the electric field from the scalar electric potential in a variety of geometries.

5.4 I can apply the electric field boundary conditions to calculate electric fields on both sides of a material boundary.

Gauss' Law

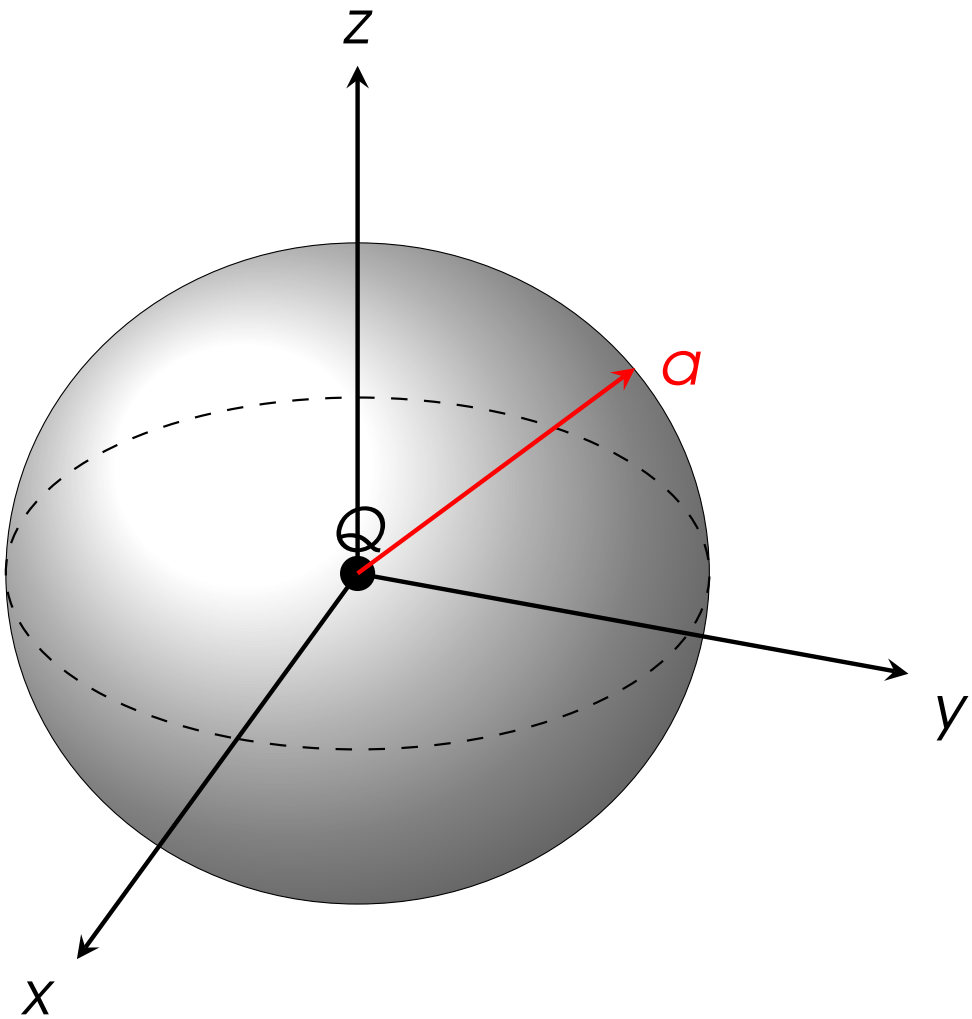
Definition

The flux of the electric charge through a **closed** surface is equal to the enclosed charge.

Integral & Differential (Point) Forms

$$\oiint_S \vec{D} \cdot d\vec{S} = Q_{\text{enclosed}} \qquad \nabla \cdot \vec{D} = \rho_V$$

Gauss' Law: General Method



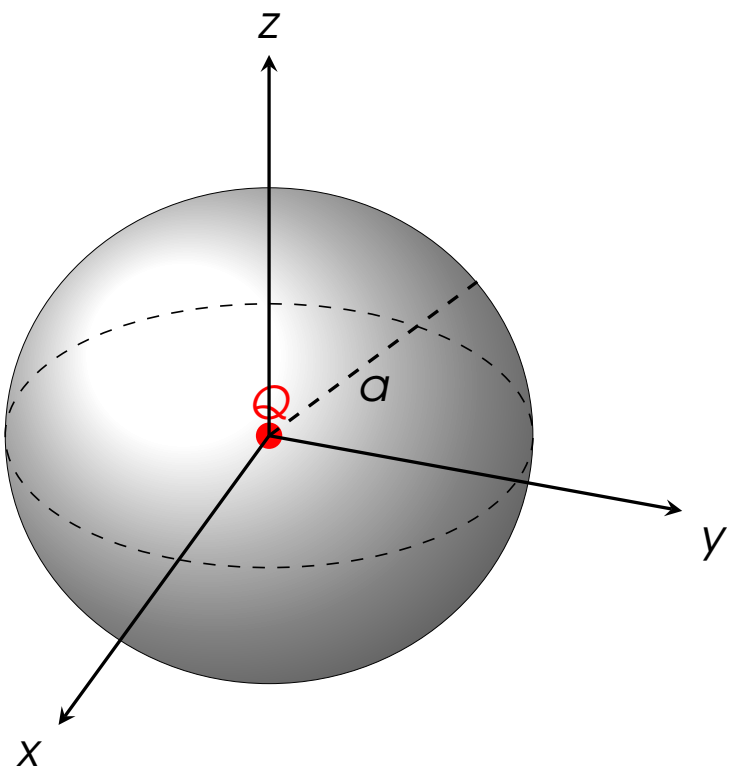
Method

1. Draw Gaussian Surface ($\vec{dS}$)
2. Determine symmetry
3. Solve Left Integral:

$$\oiint_S \varepsilon \vec{E} \cdot \vec{dS}$$

4. Determine charge enclosed, Q_{encl}

Gauss' Law: Point Charge



$$\oiint_S \vec{D} = Q_{\text{enclosed}}$$

$$\Longrightarrow \quad \vec{dS} = \hat{r} r^2 \sin(\theta)\, d\theta d\phi$$

$$\Longrightarrow \quad \vec{D} = \varepsilon \vec{E} = \varepsilon \left[\hat{\mathbf{r}} E_r + \hat{\boldsymbol{\theta}} E_\theta + \hat{\phi} E_\phi \right]$$

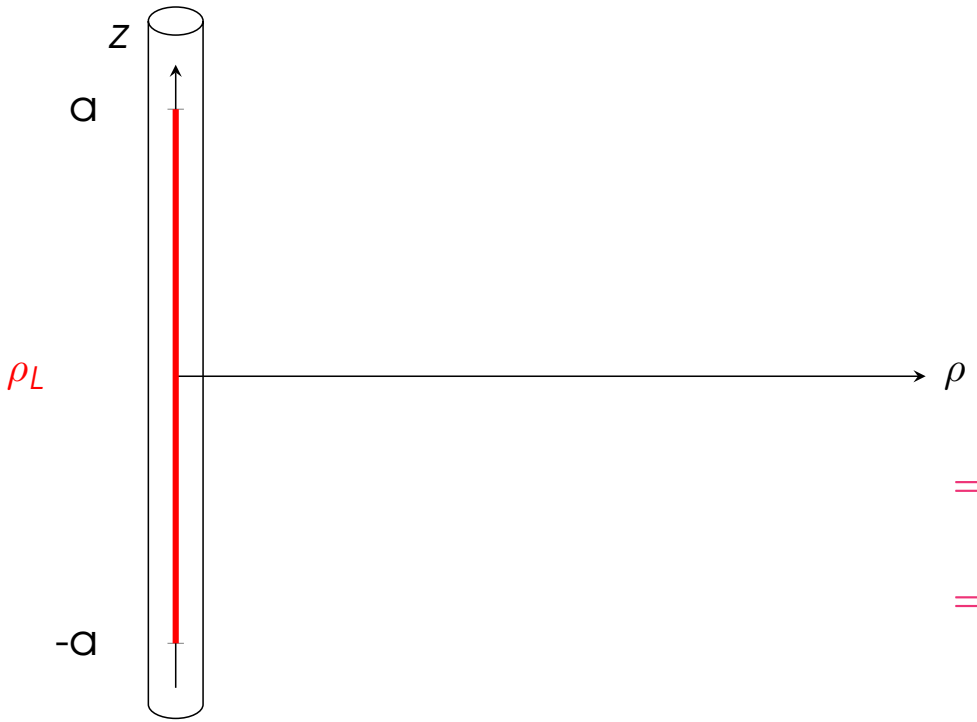
$$\vec{D} \cdot \vec{dS} = \varepsilon E_r r^2 \sin(\theta)\, d\theta d\phi$$

$$\oiint_S \vec{D} \cdot \vec{dS} = \varepsilon E_r r^2 \int_0^{2\pi} \int_0^{\pi} \sin(\theta)\, d\theta d\phi$$

$$= \varepsilon E_r r^2 (2)(2\pi)$$

$$= 4\pi \varepsilon E_r r^2$$

Gauss' Law: Line Charge



$$\oiint_S \vec{E}\cdot\vec{dS} = \frac{Q_{enclosed}}{\varepsilon}$$

$$\int_{top} \vec{E}\cdot\vec{dS} + \int_{bottom} \vec{E}\cdot\vec{dS} + \int_{side} \vec{E}\cdot\vec{dS} = \frac{Q_{enclosed}}{\varepsilon}$$

$$\Longrightarrow \vec{E} = \hat{\boldsymbol{\rho}}E_\rho + \hat{\phi}E_\phi + \hat{\mathbf{z}}E_z$$

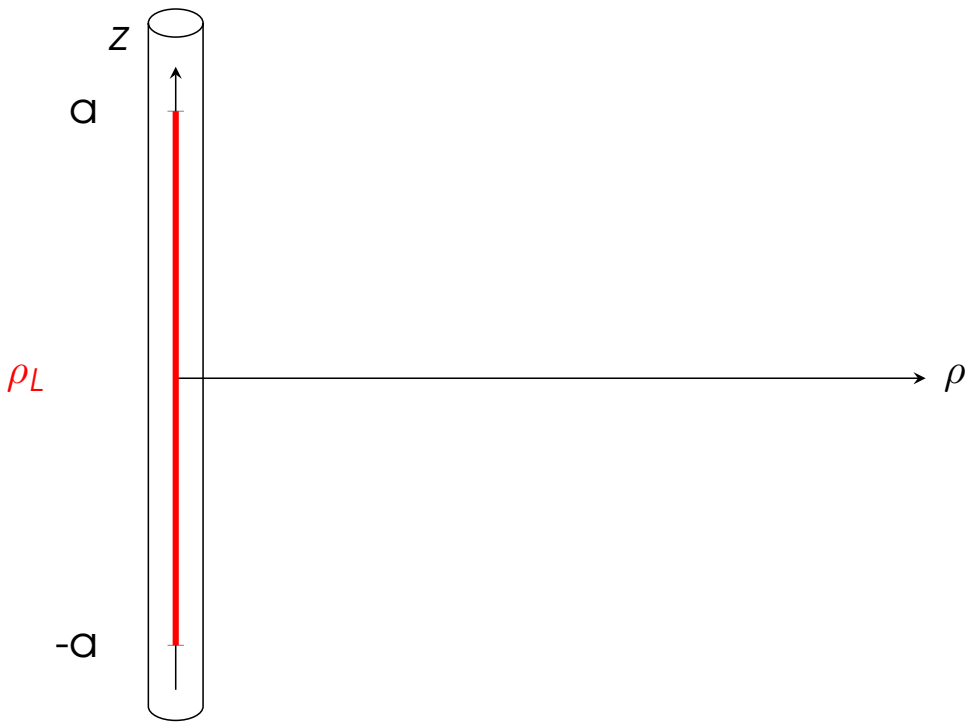
$$\Longrightarrow \vec{dS} = \hat{\boldsymbol{\rho}}\rho d\phi dz$$

$$\Longrightarrow E_\rho \int_{-a}^{a}\int_{0}^{2\pi} \rho d\phi dz = \frac{Q_{enclosed}}{\varepsilon}$$

$$E_\rho(2a)(2\pi)\rho = \frac{Q_{enclosed}}{\varepsilon}$$

$$\Longrightarrow \vec{E} = \hat{\boldsymbol{\rho}}E_\rho = \frac{Q_{enclosed}}{4a\pi\varepsilon\rho}$$

Gauss' Law: Line Charge (cont'd)



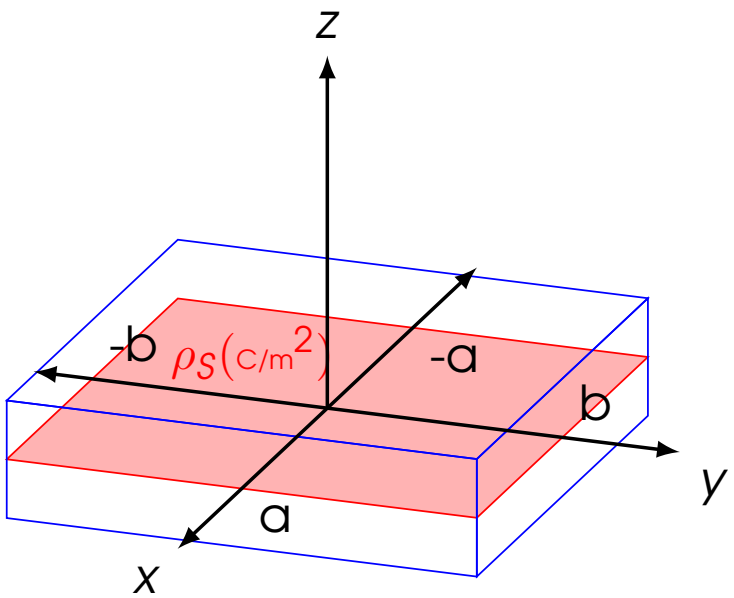
$$\vec{E} = \frac{Q_{enclosed}}{4a\pi\varepsilon\rho}$$

$$\Longrightarrow Q_{enclosed} = \int_{-a}^{a} \rho_L dz = 2a\rho_L$$

$$\Longrightarrow \vec{E} = \frac{2a\rho_L}{4a\pi\varepsilon\rho} = \frac{\rho_L}{2\pi\varepsilon\rho}$$

Gauss' Law: Sheet Charge

Find the Electric field everywhere from a sheet carrying a charge distribution of ρ_S.



$$\oiint_S \vec{E}\cdot d\vec{S} = \frac{Q_{enclosed}}{\varepsilon}$$

$$\int_{top} \vec{E}\cdot d\vec{S} + \int_{bottom} \vec{E}\cdot d\vec{S} + \int_{sides} \vec{E}\cdot d\vec{S} = \frac{Q_{enclosed}}{\varepsilon}$$

$$\text{Top} \longrightarrow d\vec{S} = \hat{\mathbf{z}}dxdy$$

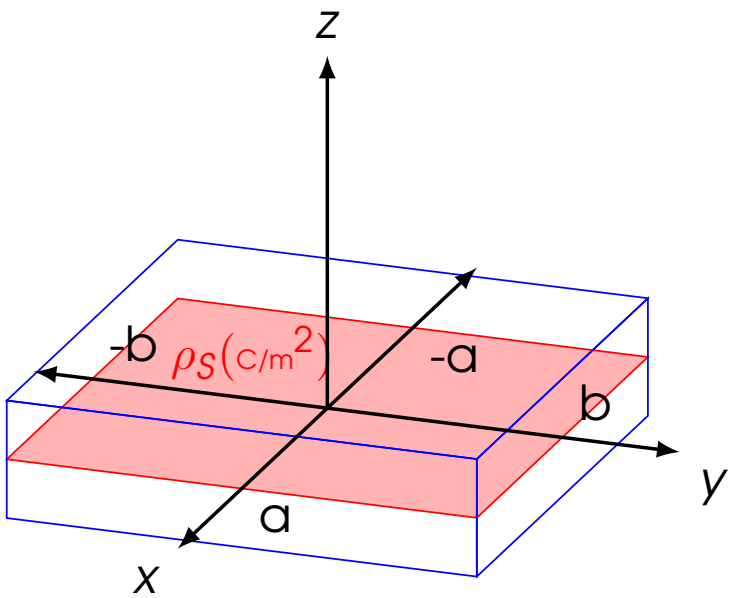
$$\Longrightarrow \int_{-b}^{b}\int_{-a}^{a} (\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y + \hat{\mathbf{z}}E_z)\cdot(\hat{\mathbf{z}}dxdy)$$

$$\text{Bottom} \longrightarrow d\vec{S} = -\hat{\mathbf{z}}dxdy$$

$$\Longrightarrow \int_{-b}^{b}\int_{-a}^{a} (\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y - \hat{\mathbf{z}}E_z)\cdot(-\hat{\mathbf{z}}dxdy)$$

Gauss' Law: Sheet Charge (cont'd)

Find the Electric field everywhere from a sheet carrying a charge distribution of ρ_S.



$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{enclosed}}}{\varepsilon}$$

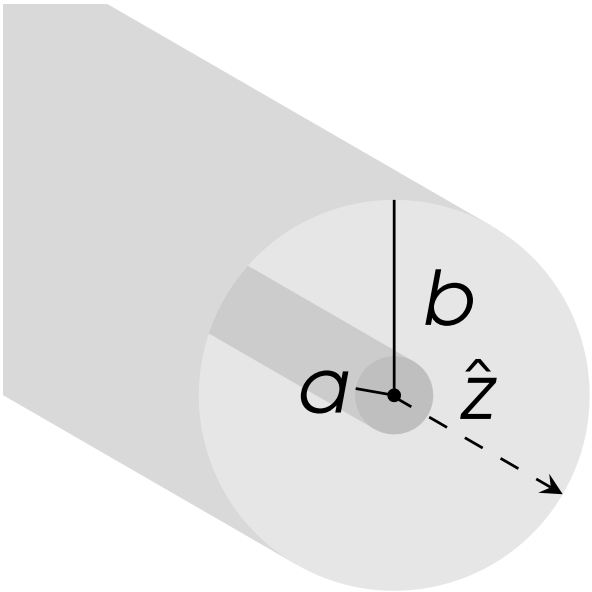
$$\Longrightarrow 2\int_{-b}^{b}\int_{-a}^{a} E_z \, dx dy = \frac{Q_{\text{enclosed}}}{\varepsilon}$$

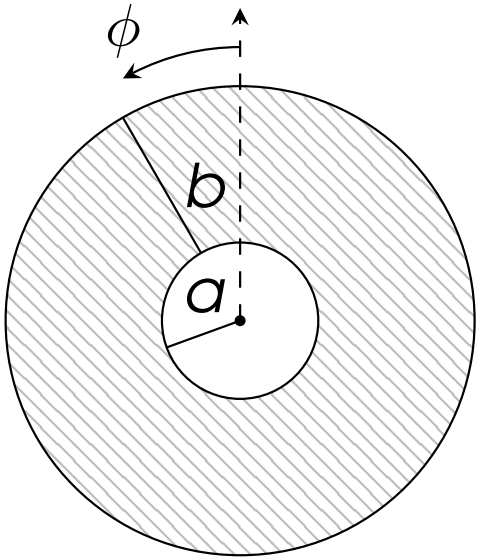
$$\Longrightarrow (2E_z)(2a)(2b) = \frac{Q_{\text{enclosed}}}{\varepsilon}$$

$$\Longrightarrow 8E_z ab = \frac{(2a)(2b)\rho_S}{\varepsilon}$$

$$\vec{E} = \hat{\mathbf{z}} E_z = \frac{\rho_S}{2\varepsilon}$$

Gauss' Law: Coaxial Cable





$$\iint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{enclosed}}}{\varepsilon}$$

$$\vec{E} = \hat{\boldsymbol{\rho}} E_\rho + \hat{\boldsymbol{\phi}} E_\phi + \hat{\mathbf{z}} E_z$$

$$d\vec{S} = \hat{\boldsymbol{\rho}} \rho dz d\phi$$

$$\Longrightarrow \int_0^{2\pi} \int_0^{L} E_\rho \rho dz d\phi = \frac{Q_{\text{enclosed}}}{\varepsilon}$$

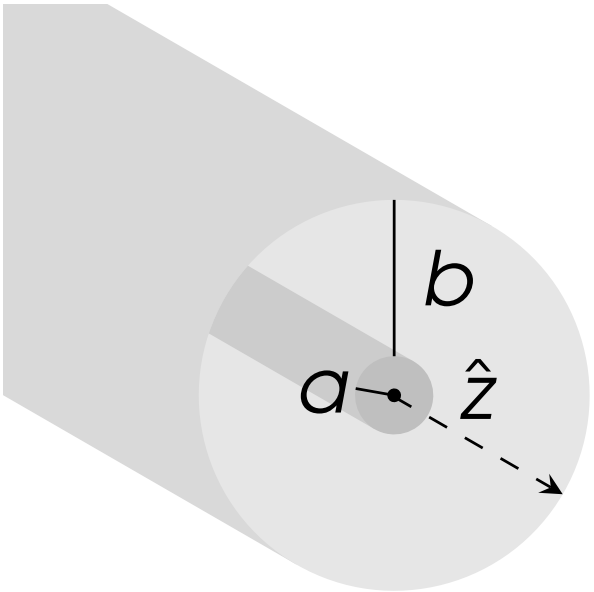
$$\Longrightarrow E_\rho \rho 2\pi L = \frac{1}{\varepsilon} \iiint \rho_V dV$$

$$\Longrightarrow \rho_V = \frac{Q_{\text{tot}}}{\text{volume}} = \frac{Q_{\text{tot}}}{\pi a^2 L}$$

$$\Longrightarrow E_\rho \rho 2\pi L = \frac{1}{\varepsilon} \iiint \frac{Q_{\text{tot}}}{\pi a^2 L} dV$$

$$\Longrightarrow E_\rho \rho 2\pi L = \frac{Q_{\text{tot}}}{\pi a^2 L \varepsilon} \int_0^{\rho_0} \rho d\rho \int_0^{2\pi} d\phi \int_0^{L} dz$$

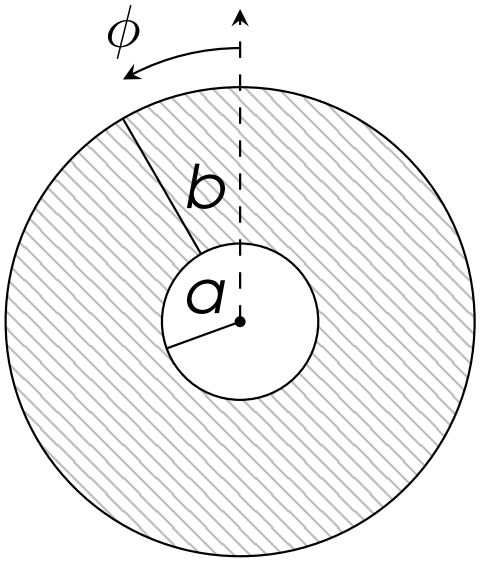
Gauss' Law: Coaxial Cable (cont'd)



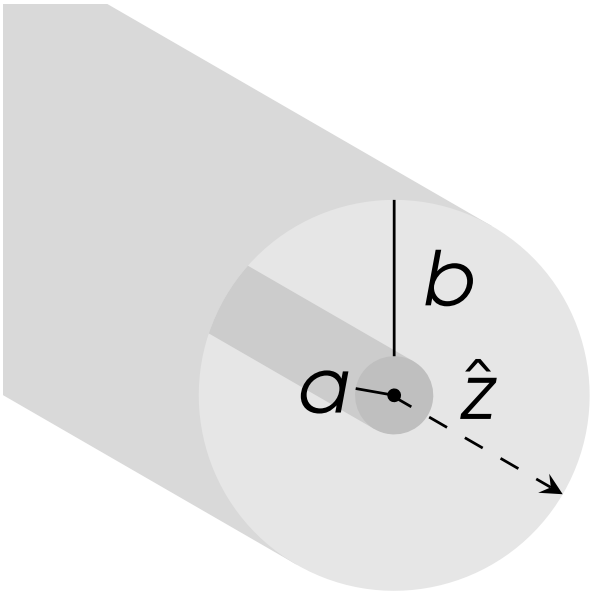
$$\oiint_S \vec{E} \cdot \vec{dS} = \frac{Q_{\text{enclosed}}}{\varepsilon}$$

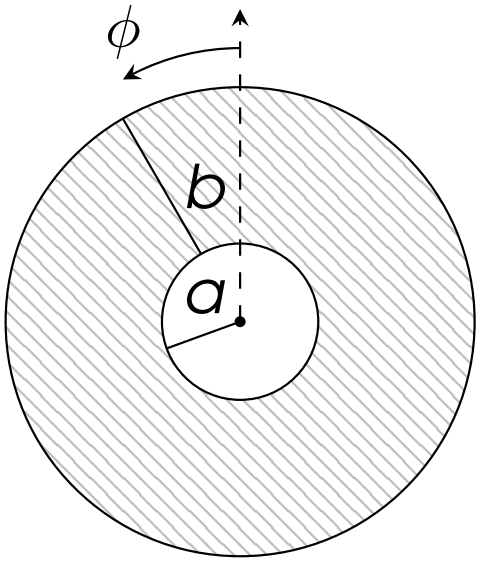
$$E_\rho \rho 2\pi L = \frac{Q_{\text{tot}}}{\pi a^2 L \varepsilon} \left(\rho_0^2\right) (2\pi) (L)$$

$$E_\rho = \frac{Q_{\text{tot}} \rho_0^2}{2\pi \varepsilon a^2 L \rho}$$



Gauss' Law: Coaxial Cable (cont'd)





Gauss' Law: Point (differential) Form

Derivation

$$\oiint \vec{D} \cdot d\vec{S} = Q_{\text{encl}}$$

$$\iiint_V \nabla \cdot \vec{D} dV = \iiint_V \rho_V dV$$

$$\nabla \cdot \vec{D} = \rho_V$$

$$\nabla \cdot \vec{E} = \frac{\rho_V}{\varepsilon}$$

EXAMPLE

Given the following static electric field in free space, find the charge density (ρ) at the point $(2, -2, 0)$

$$\vec{E}(\vec{r}) = \hat{\mathbf{x}}Ax^2 + \hat{\mathbf{y}}Bz + \hat{\mathbf{z}}Cx^2z \longrightarrow A = 3\text{V}/\text{m}^3 \quad B = 2\text{V}/\text{m}^3 \quad C = 1\text{V}/\text{m}^4$$

Solution

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\varepsilon_0 \left[\frac{\partial}{\partial x}\left(3x^2\right) + \frac{\partial}{\partial y}\left(2z\right) + \frac{\partial}{\partial z}\left(x^2z\right) \right] = \rho$$

$$\varepsilon_0 \left(6x + 0 + x^2\right) = \rho$$

$$8.854 \times 10^{-12}\text{F/m}\left[(6)(2) + 2^2\right] = \rho$$

$$141.7\text{pC/m}^3 = \rho$$